# PLURICANONICAL SYSTEMS OF PROJECTIVE VARIETIES OF GENERAL TYPE II

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**Abstract**

We prove that there exists a positive integer $\nu_n$ depending only on $n$ such that for every smooth projective $n$-fold of general type $X$ defined over complex numbers, $\mid mK_X \mid$ gives a birational rational map from $X$ into a projective space for every $m \geq \nu_n$. This theorem gives an affirmative answer to Severi's conjecture. MSC: 14J40, 32J18.

## Contents



# 1 Introduction

Let $X$ be a smooth projective variety and let $K_X$ be the canonical bundle of $X$. $X$ is said to be a general type, if there exists a positive integer $m$ such that the pluricanonical system $\mid mK_X \mid$ gives a birational (rational) embedding of $X$. The following problem is fundamental to study projective varieties of general type.

**Problem** Find a positive integer $\nu_n$ depending only on $n$ such that for every smooth projective $n$-fold $X$, $\mid mK_X \mid$ gives a birational rational map from $X$ into a projective space for every $m \geqq \nu_n$. $\square$

If $X$ is a projective curve of genus $\geqq 2$, it is well known that $\mid 3K_X \mid$ gives a projective embedding. In the case that $X$ is a smooth projective surface of general type, E. Bombieri showed that $\mid 5K_X \mid$ gives a birational rational map from $X$ into a projective space ([3]). But for the case of $\dim X \geqq 3$, very little is known about the above problem.

The main purpose of this article is to prove the following theorems in full generality.

**Theorem 1.1** *There exists a positive integer $\nu_n$ which depends only on $n$ such that for every smooth projective $n$-fold $X$ of general type defined over complex numbers, $\mid mK_X \mid$ gives a birational rational map from $X$ into a projective space for every $m \geqq \nu_n$.* $\square$

Theorem 1.1 is very much related to the theory of minimal models. It has been conjectured that for every nonuniruled smooth projective variety $X$, there exists a projective variety $X_{min}$ such that

1. $X_{min}$ is birationally equivalent to $X$,
2. $X_{min}$ has only **Q**-factorial terminal singularities,
3. $K_{X_{min}}$ is a nef **Q**-Cartier divisor.

$X_{min}$ is called a minimal model of $X$. To construct a minimal model, the minimal model program (MMP) has been proposed (cf. [15, p.96]). The minimal model program was completed in the case of 3-folds by S. Mori ([19]).

The proof of Theorem 1.1 can be very much simplified, if we assume the existence of minimal models for projective varieties of general type ([34]). The proof here is modeled after the proof under the existence of minimal models by using the theory of AZD originated by the author ([26, 27]).

The major difficulty of the proof of Theorem 1.1 is to find **"a (universal) lower bound" of the positivity of** $K_X$. In fact Theorem 1.1 is equivalent to the following theorem.

**Theorem 1.2** *There exists a positive number $C_n$ which depends only on $n$ such that for every smooth projective $n$-fold $X$ of general type defined over complex numbers,*

$$\mu(X, K_X) := n! \cdot \overline{\lim}_{m\to\infty} m^{-\dim X} \dim H^0(X, \mathcal{O}_X(mK_X)) \geqq C_n$$

*holds.* $\square$

We note that $\mu(X, K_X)$ is equal to the intersection number $K_X^n$ for a minimal projective $n$-fold $X$ of general type. In Theorems 1.1 and 1.2, the numbers $\nu_n$ and $C_n$ have not yet been computed effectively.

The relation between Theorems 1.1 and 1.2 is as follows. Theorem 1.2 means that there exists a universal lower bound of the positivity of canonical bundle

of smooth projective variety of general type with a fixed dimension. On the other hand, for a smooth projective variety of general type $X$, let us consider the lower bound of $m$ such that $\mid mK_X \mid$ gives a birational embedding. Such a lower bound depends on the positivity of $K_X$ on certain subvarieties which appear as the strata of the filtrations as in [30, 1](cf. Section 3.1).

The positivity of $K_X$ on the subvarieites can be related to the positivity of the canonical bundles of the smooth models of the subvarieties via the subadjunction theorem due to Kawamata ([11]). We note that for a smooth projective variety $X$ of general type there exists a nonempty open subset $U_0$ in countable Zariski topology such that for every $x \in U_0$, any subvariety containing $x$ is of general type.

The organization of the paper is as follows. In Section 2, we review the basic techniques to prove Theorems 1.1 and 1.2.

In Section 3, we prove Theorems 1.1 and 1.2 without assuming the existence of minimal models for projective varieties of general type. Here we use the AZD (cf. Section 2.2) of $K_X$ instead of minimal models. And we use the subadjunction theorem (Theorem 2.22) and the positivity theorem (Theorem 2.28) due to Kawamata.

In Section 4, we discuss the application of Theorems 1.1 and 1.2 to Severi-Iitaka's conjecture.

In this paper all the varieties are defined over $\mathbf{C}$.

This is the continuation of the paper [34] and is a transcription of the latter half of [33].

The author would like to express his sincere thanks to Professor Akira Fujiki who helped him to improve the exposition.

# 2 Preliminaries

In this section, we shall summerize the basic analyic tools to prove Theorems 1.1 and 1.2 by transcripting the proof of Theorems 1.1 and 1.2 assuming MMP ([34]).

## 2.1 Multiplier ideal sheaves and singularities of divisors

In this subsection, we shall review the relation between multiplier ideal sheaves and singularities of divisors. Throughout this subsection $L$ will denote a holomorphic line bundle on a complex manifold $M$.

**Definition 2.1** *A singular hermitian metric $h$ on $L$ is given by*

$$h = e^{-\varphi} \cdot h_0,$$

*where $h_0$ is a $C^\infty$-hermitian metric on $L$ and $\varphi \in L^1_{loc}(M)$ is an arbitrary function on $M$. We call $\varphi$ the weight function of $h$ with respect to $h_0$.* $\square$

The curvature current $\Theta_h$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where $\partial\bar{\partial}$ is taken in the sense of a current. The $L^2$-sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\mathcal{L}^2(L,h)(U) := \{\sigma \in \Gamma(U, \mathcal{O}_M(L)) \mid h(\sigma,\sigma) \in L^1_{loc}(U)\},$$

where $U$ runs over the open subsets of $M$. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L,h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the **multiplier ideal sheaf** of $(L,h)$. If we write $h$ as

$$h = e^{-\varphi} \cdot h_0,$$

where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{loc}(M)$ is the weight function, we see that

$$\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$$

holds. For $\varphi \in L^1_{loc}(M)$ we define the multiplier ideal sheaf of $\varphi$ by

$$\mathcal{I}(\varphi) := \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi}).$$

**Example 2.2** *Let $m$ be a positive integer. Let $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$ be the global section. Then*

$$h := \frac{1}{\mid \sigma \mid^{2/m}} = \frac{h_0}{(h_0^m(\sigma,\sigma))^{1/m}}$$

*is a singular hemitian metric on $L$, where $h_0$ is an arbitrary $C^\infty$-hermitian metric on $L$ (the righthand side is ovbiously independent of $h_0$). The curvature $\Theta_h$ is given by*

$$\Theta_h = \frac{2\pi\sqrt{-1}}{m}(\sigma),$$

*where $(\sigma)$ denotes the current of integration over the divisor of $\sigma$.* $\square$

**Definition 2.3** *$L$ is said to be pseudoeffective, if there exists a singular hermitian metric $h$ on $L$ such that the curvature current $\Theta_h$ is a closed positive current.*

*Also a singular hermitian line bundle $(L,h)$ is said to be pseudoeffective, if the curvature current $\Theta_h$ is a closed positive current.* $\square$

Let $m$ be a positive integer and $\{\sigma_i\}$ a finite number of global holomorphic sections of $mL$. Let $\phi$ be a $C^\infty$-function on $M$. Then

$$h := e^{-\phi} \cdot \frac{1}{\sum_i \mid \sigma_i \mid^{2/m}}$$

defines a singular hermitian metric on $L$. We call such a metric $h$ a singular hermitian metric on $L$ with **algebraic singularities**. Singular hermitian metrics with algebraic singularities are particulary easy to handle, because its multiplier ideal sheaf of the metric can be controlled by taking a resolution of the base scheme $\cap_i(\sigma_i)$.

Let $D = \sum a_i D_i$ be an effective $\mathbf{Q}$-divisor on $X$. Let $\sigma_i$ be a section of $\mathcal{O}_X(D_i)$ with divisor $D_i$ respectively. Then we define

$$\mathcal{I}(D) := \mathcal{I}(\sum_i a_i \log h_i(\sigma_i,\sigma_i))$$

and call it the multiplier ideal sheaf of the divisor $D$, where $h_i$ denotes a $C^\infty$-hermitian metric of $\mathcal{O}_X(D_i)$ respectively. It is clear that $\mathcal{I}(D)$ is independent of the choice of the hermitian metrics $\{h_i\}$.

Let us consider the relation between $\mathcal{I}(D)$ and singularities of $D$.

**Definition 2.4** *Let $X$ be a normal variety and $D = \sum_i d_i D_i$ an effective* $\mathbf{Q}$-*divisor such that $K_X + D$ is* $\mathbf{Q}$-*Cartier. If $\mu : Y \longrightarrow X$ is a log resolution of the pair $(X, D)$, i.e., $\mu$ is a composition of successive blowing ups with smooth centers such that $Y$ is smooth and the support of $f^*D$ is a divisor with normal crossings, then we can write*

$$K_Y + \mu_*^{-1}D = \mu^*(K_X + D) + F$$

*with $F = \sum_j e_j E_j$ for the exceptional divisors $\{E_j\}$, where $\mu_*^{-1}D$ denotes the strict transform of $D$. We call $F$ the discrepancy and $e_j \in \mathbf{Q}$ the discrepancy coefficient for $E_j$. We regard $-d_i$ as the discrepancy coefficient of $D_i$.*

*The pair $(X, D)$ is said to have only* **Kawamata log terminal singularities** *(***KLT***)(resp.* **log canonical singularities***(***LC***)), if $d_i < 1$(resp. $\leqq 1$) for all $i$ and $e_j > -1$ (resp. $\geqq -1$) for all $j$ for a log resolution $\mu : Y \longrightarrow X$. One can also say that $(X, D)$ is KLT (resp. LC), or $K_X + D$ is KLT (resp. LC), when $(X, D)$ has only KLT (resp. LC). The pair $(X, D)$ is said to be KLT (resp. LC) at a point $x_0 \in X$, if $(U, D\mid_U)$ is KLT (resp. LC) for some neighbourhood $U$ of $x_0$.* $\square$

The following proposition is a dictionary between algebraic geometry and the $L^2$-method.

**Proposition 2.5** *Let $D$ be a divisor on a smooth projective variety $X$. Then $(X, D)$ is KLT, if and only if $\mathcal{I}(D)$ is trivial ($= \mathcal{O}_X$).* $\square$

The proof is trivial and left to the reader. To locate the co-support of the multiplier ideal the following notion is useful.

**Definition 2.6** *A subvariety $W$ of $X$ is said to be a* **center of log canonical singularities** *for the pair $(X, D)$, if there is a birational morphism from a normal variety $\mu : Y \longrightarrow X$ and a prime divisor $E$ on $Y$ with the discrepancy coefficient $e \leqq -1$ such that $\mu(E) = W$.* $\square$

The set of all the centers of log canonical singularities is denoted by $CLC(X, D)$. For a point $x_0 \in X$, we define $CLC(X, x_0, D) := \{W \in CLC(X, D) \mid x_0 \in W\}$. We quote the following proposition to introduce the notion of the minimal center of logcanoical singularities.

**Proposition 2.7** *([12, p.494, Proposition 1.5]) Let $X$ be a normal variety and $D$ an effective* $\mathbf{Q}$-*Cartier divisor such that $K_X + D$ is* $\mathbf{Q}$-*Cartier. Assume that $X$ is KLT and $(X, D)$ is LC. If $W_1, W_2 \in CLC(X, D)$ and $W$ an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. This implies that if $(X, D)$ is LC but not KLT, then there exists the unique minimal element of $CLC(X, D)$. Also if $(X, D)$ is LC but not KLT at a point $x_0 \in X$, then there exists the unique minimal element of $CLC(X, x_0, D)$.*

We call these minimal elements the **minimal center of LC singularities** of $(X, D)$ and the **minimal center of LC singularities of** $(X, D)$ **at** $x_0$ respectively.

## 2.2 Analytic Zariski decomposition

To study a pseudoeffective line bundle we introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle a pseudoeffective line bundle, as if it were a nef line bundle.

**Definition 2.8** *Let $M$ be a compact complex manifold and let $L$ be a line bundle on $M$. A singular hermitian metric $h$ on $L$ is said to be an* **analytic Zariski decomposition** *(***AZD** *in short), if the followings hold.*

1. *$\Theta_h$ is a closed positive current,*
2. *for every $m \geqq 0$, the natural inclusion*
$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL))$$
*is isomorphim.*

$\square$

**Remark 2.9** *If an AZD exists on a line bundle $L$ on a smooth projective variety $M$, $L$ is pseudoeffective by the condition 1 above.* $\square$

**Theorem 2.10** *([26, 27]) Let $L$ be a big line bundle on a smooth projective variety $M$. Then $L$ has an AZD.* $\square$

As for the existence for general pseudoeffective line bundles, now we have the following theorem.

**Theorem 2.11** *([6, Theorem 1.5]) Let $X$ be a smooth projective variety and let $L$ be a pseudoeffective line bundle on $X$. Then $L$ has an AZD.* $\square$

Although the proof is in [6], we shall give a proof here, because we shall use it afterward.

Let $h_0$ be a fixed $C^\infty$-hermitian metric on $L$. Let $E$ be the set of singular hermitian metric on $L$ defined by

$$E = \{h; h : \text{lowersemicontinuous singular hermitian metric on } L,$$
$$\Theta_h \text{ is positive}, \frac{h}{h_0} \geq 1\}.$$

Since $L$ is pseudoeffective, $E$ is nonempty. We set

$$h_L = h_0 \cdot \inf_{h \in E} \frac{h}{h_0},$$

where the infimum is taken pointwise. The supremum of a family of plurisubharmonic functions uniformly bounded from above is known to be again plurisubharmonic, if we modify the supremum on a set of measure 0(i.e., if we take the uppersemicontinuous envelope) by the following theorem of P. Lelong.

**Theorem 2.12** *([17, p.26, Theorem 5]) Let $\{\varphi_t\}_{t \in T}$ be a family of plurisubharmonic functions on a domain $\Omega$ which is uniformly bounded from above on every compact subset of $\Omega$. Then $\psi = \sup_{t \in T} \varphi_t$ has a minimum uppersemicontinuous majorant $\psi^*$ which is plurisubharmonic. We call $\psi^*$ the uppersemicontinuous envelope of $\psi$.* $\square$

**Remark 2.13** *In the above theorem the equality* $\psi = \psi^*$ *holds outside of a set of measure* 0*(cf.[17, p.29]).* $\square$

By Theorem 2.12, we see that $h_L$ is also a singular hermitian metric on $L$ with $\Theta_{h_L} \geq 0$. Suppose that there exists a nontrivial section $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$ for some $m$ (otherwise the second condition in Definition 2.8 is empty). We note that

$$\frac{1}{\mid \sigma \mid^{\frac{2}{m}}}$$

gives the weight of a singular hermitian metric on $L$ with curvature $2\pi m^{-1}(\sigma)$, where $(\sigma)$ is the current of integration along the zero set of $\sigma$. By the construction we see that there exists a positive constant $c$ such that

$$(*) \qquad \frac{h_0}{\mid \sigma \mid^{\frac{2}{m}}} \geq c \cdot h_L$$

holds. Hence

$$\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m))$$

holds. In praticular

$$\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h_L^m))$$

holds. This means that $h_L$ is an AZD of $L$. $\square$

**Remark 2.14** *By the above proof (see $(*)$) we have that for the AZD $h_L$ constructed as above*

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m)) \simeq H^0(X, \mathcal{O}_X(mL))$$

*holds for every $m$, where $\mathcal{I}_\infty(h_L^m)$ denotes the $L^\infty$-multiplier ideal sheaf, i.e., for every open subset $U$ in $X$,*

$$\mathcal{I}_\infty(h_L^m)(U) := \{f \in \mathcal{O}_X(U) \mid\mid f \mid^2 (h_L/h_0)^m \in L^\infty_{loc}(U)\},$$

*where $h_0$ is a $C^\infty$-hermitian metric on $L$.* $\square$

Entirely the same proof as that of Theorem 2.11, we obtain the following corollary.

**Corollary 2.15** *Let $(L, h_0)$ be a singular hermitian line bundle on a compact Kähler manifold $(X, \omega)$. Suppose that*

$$E(L, h_0) := \{\varphi \in L^1_{loc}(X) \mid \varphi \leqq 0,\ \Theta_{h_0} + \sqrt{-1}\partial\bar{\partial}\varphi \geqq 0\}$$

*is nonempty. Then if we define the function $\varphi_P \in L^1_{loc}(X)$ by*

$$\varphi_P(x) := \sup\{\varphi(x) \mid \varphi \in E\} \quad (x \in X).$$

*Then $h := e^{-\varphi_P} \cdot h_0$ is a singular hermitian metric on $L$ such that*

1. $\Theta_h \geqq 0$.

*2.* $H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h^m)) \simeq H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_0^m))$ *holds for every* $m \geqq 0$. $\square$

We call $h$ an AZD of $(L, h_0)$. This is a slight generalization of the notion of AZD's of pseudoeffective line bundles.

**Remark 2.16** *In Corollary 2.15,* $E(L, h_0)$ *is nonempty, if there exists a positive integer* $m_0$ *and* $\sigma \in H^0(X, \mathcal{O}_X(m_0 L) \otimes \mathcal{I}_\infty(h_0^{m_0}))$ *such that* $h_0^{m_0}(\sigma, \sigma) \leqq 1$. *In this case*

$$\varphi := \frac{1}{m_0} \log h_0^{m_0}(\sigma, \sigma)$$

*belongs to* $E(L, h_0)$.

## 2.3 The $L^2$-extension theorem

Let $M$ be a complex manifold of dimension $n$ and let $S$ be a closed complex submanifold of $M$. Then we consider a class of continuous function $\Psi : M \longrightarrow [-\infty, 0)$ such that

1. $\Psi^{-1}(-\infty) \supset S$,
2. if $S$ is $k$-dimensional around a point $x$, there exists a local coordinate system $(z_1, \ldots, z_n)$ on a neighbourhood of $x$ such that $z_{k+1} = \cdots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \backslash S} \mid \Psi(z) - (n-k) \log \sum_{j=k+1}^{n} \mid z_j \mid^2 \mid < \infty.$$

The set of such functions $\Psi$ will be denoted by $\sharp(S)$.

For each $\Psi \in \sharp(S)$, one can associate a positive measure $dV_M[\Psi]$ on $S$ as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_k} f \, d\mu \geqq \overline{\lim}_{t \to \infty} \frac{2(n-k)}{v_{2n-2k-1}} \int_M f \cdot e^{-\Psi} \cdot \chi_{R(\Psi,t)} dV_M$$

for any nonnegative continuous function $f$ with $\operatorname{supp} f \subset\subset M$. Here $S_k$ denotes the $k$-dimensional component of $S$, $v_m$ denotes the volume of the unit sphere in $\mathbf{R}^{m+1}$ and $\chi_{R(\Psi,t)}$ denotes the characteristic funciton of the set

$$R(\Psi, t) = \{x \in M \mid -t-1 < \Psi(x) < -t\}.$$

Let $M$ be a complex manifold and let $(E, h_E)$ be a holomorphic hermitian vector bundle over $M$. Given a positive measure $d\mu_M$ on $M$, we shall denote $A^2(M, E, h_E, d\mu_M)$ the space of $L^2$ holomorphic sections of $E$ over $M$ with respect to $h_E$ and $d\mu_M$. Let $S$ be a closed complex submanifold of $M$ and let $d\mu_S$ be a positive measure on $S$. The measured submanifold $(S, d\mu_S)$ is said to be a set of interpolation for $(E, h_E, d\mu_M)$, or for the sapce $A^2(M, E, h_E, d\mu_M)$, if there exists a bounded linear operator

$$I : A^2(S, E \mid_S, h_E, d\mu_S) \longrightarrow A^2(M, E, h_E, d\mu_M)$$

such that $I(f) \mid_S = f$ for any $f \in A^2(S, E \mid_S, h_E, d\mu_S)$. $I$ is called an interpolation operator. The following theorem is crucial.

**Theorem 2.17** *([23, Theorem 4]) Let $M$ be a complex manifold with a continuous volume form $dV_M$, let $E$ be a holomorphic vector bundle over $M$ with $C^\infty$-fiber metric $h_E$, let $S$ be a closed complex submanifold of $M$, let $\Psi \in \sharp(S)$ and let $K_M$ be the canonical bundle of $M$. Then $(S, dV_M(\Psi))$ is a set of interpolation for $(E \otimes K_M, h_E \otimes (dV_M)^{-1}, dV_M)$, if the followings are satisfied.*

1. *There exists a closed set $X \subset M$ such that*
   (a) *$X$ is locally negligble with respect to $L^2$-holomorphic functions, i.e., for any local coordinate neighbourhood $U \subset M$ and for any $L^2$-holomorphic function $f$ on $U \backslash X$, there exists a holomorphic function $\tilde{f}$ on $U$ such that $\tilde{f} \mid U \backslash X = f$.*
   (b) *$M \backslash X$ is a Stein manifold which intersects with every component of $S$.*
2. *$\Theta_{h_E} \geqq 0$ in the sense of Nakano,*
3. *$\Psi \in \sharp(S) \cap C^\infty(M \backslash S)$,*
4. *$e^{-(1+\epsilon)\Psi} \cdot h_E$ has semipositive curvature in the sense of Nakano for every $\epsilon \in [0, \delta]$ for some $\delta > 0$.*

*Under these conditions, there exists a constant $C$ and an interpolation operator from $A^2(S, E \otimes K_M \mid_S, h \otimes (dV_M)^{-1} \mid_S, dV_M[\Psi])$ to $A^2(M, E \otimes K_M, h \otimes (dV_M)^{-1}.dV_M)$ whose norm does not exceed $C \cdot \delta^{-3/2}$. If $\Psi$ is plurisubharmonic, the interpolation operator can be chosen so that its norm is less than $2^4\pi^{1/2}$.* $\square$

The above theorem can be generalized to the case that $(E, h_E)$ is a singular hermitian line bundle with semipositive curvature current (we call such a singular hermitian line bundle $(E, h_E)$ a **pseudoeffective singular hermitian line bundle**) as was remarked in [23].

**Lemma 2.18** *Let $M, S, \Psi, dV_M, dV_M[\Psi], (E, h_E)$ be as in Theorem 2.17. Let $(L, h_L)$ be a pseudoeffective singular hermitian line bundle on $M$. Then $(S, dV_M[\Psi])$ is a set of interpolation for $(K_M \otimes E \otimes L, dV_M^{-1} \otimes h_E \otimes h_L)$.* $\square$

## 2.4 A construction of the function $\Psi$

Here we shall show the standard construction of the function $\Psi$ in Theorem 2.17. Let $M$ be a smooth projective $n$-fold and let $S$ be a $k$-dimensional (not necessary smooth) subvariety of $M$. Let $\mathcal{U} = \{U_\gamma\}$ be a finite Stein covering of $M$ and let $\{f_1^{(\gamma)}, \ldots, f_{m(\gamma)}^{(\gamma)}\}$ be a generator of the ideal sheaf associated with $S$ on $U_\gamma$. Let $\{\phi_\gamma\}$ be a partition of unity which subordinates to $\mathcal{U}$. We set

$$\Psi := (n-k)\sum_\gamma \phi_\gamma \cdot (\sum_{\ell=1}^{m(\gamma)} \mid f_\ell^{(\gamma)} \mid^2).$$

Then the residue volume form $dV[\Psi]$ is defined as in the last subsection. Here the residue volume form $dV[\Psi]$ of a continuous volume form $dV$ on $M$ is not well defined on the singular locus of $S$. But this is not a difficulty to apply Theorem 2.17 or Lemma 2.18, since there exists a proper Zariski closed subset $Y$ of $X$ such that $(X - Y) \cap S$ is smooth.

## 2.5 Volume of pseudoeffective line bundles

To measure the positivity of big line bundles on a projective variety, we shall introduce the notion of volume of a projective variety with respect to a big line bundle.

**Definition 2.19** *Let $L$ be a line bundle on a compact complex manifold $M$ of dimension $n$. We define the* **volume** *of $M$ with respect to $L$ by*

$$\mu(M, L) := n! \cdot \overline{\lim}_{m\to\infty} m^{-n} \dim H^0(M, \mathcal{O}_M(mL)).$$

$\square$

With respect to a pseudoeffective singular hermitian line bundle (for the definition of pseudoeffective singular hermitian line bundles, see the last part of Section 2.3), we define the volume as follows.

**Definition 2.20** *([28]) Let $(L, h)$ be a pseudoeffective singular hermitian line bundle on a smooth projective variety $X$ of dimension $n$. We define the* **volume of $X$ with respect to** $(L, h)$ *by*

$$\mu(X, (L, h)) := n! \cdot \overline{\lim}_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)).$$

*A pseudoeffective singular hermitian line bundle $(L, h)$ is said to be big, if $\mu(X, (L, h)) > 0$ holds.*

*We may consider $\mu(X, (L, h))$ as the* **intersection number** *$(L, h)^n$. We also denote $\mu(X, (L, h))$ by $(L, h)^n$. Let $Y$ be a subvariety of $X$ of dimension $d$ and let $\pi_Y : \tilde{Y} \longrightarrow Y$ be a resolution of $Y$. We define $\mu(Y, (L, h) \mid_Y)$ as*

$$\mu(Y, (L, h) \mid_Y) := \mu(\tilde{Y}, \pi_Y^*(L, h)).$$

*The righthand side is independent of the choice of the resolution $\pi$ because of the remark below. We also denote $\mu(Y, (L, h) \mid_Y)$ by $(L, h)^d \cdot Y$.* $\square$

**Remark 2.21** *Let us use the same notations in Definition 2.20. Let $\pi : \tilde{X} \longrightarrow X$ be any modification. Then*

$$\mu(X, (L, h)) = \mu(\tilde{X}, \pi^*(L, h))$$

*holds, since*

$$\pi_*(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}) \otimes \mathcal{I}(\pi^* h^m)) = \mathcal{O}_X(K_X) \otimes \mathcal{I}(h^m)$$

*holds for every $m$ and*

$$\overline{\lim}_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)) = \overline{\lim}_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mL + D) \otimes \mathcal{I}(h^m))$$

*holds for any Cartier divisor $D$ on $X$. This last equality can be easily checked, if $D$ is a smooth irreducible divisor, by using the exact sequence*

$$0 \to \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m) \to \mathcal{O}_X(mL + D) \otimes \mathcal{I}(h^m) \to \mathcal{O}_D(mL + D) \otimes \mathcal{I}(h^m) \to 0.$$

*For a general $D$, the equality follows by expressing $D$ as a difference of two very ample divisors.* $\square$

## 2.6 A subadjunction theorem

Let $M$ be a smooth projective variety and let $(L, h_L)$ be a singular hermitian line bundle on $M$ such that $\Theta_{h_L} \geqq 0$ on $M$. We assume that $h_L$ is lowersemicontinuous. This is a technical assumption so that a local potential of the curvature current of $h$ is plurisubharmonic.

Let $m_0$ be a positive integer. Let $\sigma \in \Gamma(M, \mathcal{O}_M(m_0L) \otimes \mathcal{I}(h))$ be a global section. Let $\alpha$ be a positive rational number $\leqq 1$ and let $S$ be an irreducible subvariety of $M$ such that $(M, \alpha(\sigma))$ is LC(log canonical) but not KLT(Kawamata log terminal) on the generic point of $S$ and $(M, (\alpha - \epsilon)(\sigma))$ is KLT on the generic point of $S$ for every $0 < \epsilon << 1$. We set

$$\Psi_S = \alpha \log h_L(\sigma, \sigma).$$

Suppose that $S$ is smooth for simplicity (if $S$ is not smooth, we just need to take an embedded resolution to apply Theorems 2.22, 2.23 below). We shall assume that $S$ is not contained in the singular locus of $h_L$, where the singular locus of $h_L$ means the set of points where $h$ is $+\infty$. Let $dV$ be a $C^\infty$-volume form on $M$.

Then as in Section 2.3, we may define a (possibly singular) measure $dV[\Psi_S]$ on $S$. This can be viewed as follows. Let $f : N \longrightarrow M$ be a log resolution of $(X, \alpha(\sigma))$. Then as in Section 2.4, we may define the singular volume form $f^*dV[f^*\Psi_S]$ on the divisorial component of $f^{-1}(S)$ (the volume form is identically 0 on the components with discrepancy $> -1$). The singular volume form $dV[\Psi_S]$ is defined as the fibre integral of $f^*dV[f^*\Psi_S]$ (the actual integration takes place only on the components with discrepancy $-1$). Let $d\mu_S$ be a $C^\infty$-volume form on $S$ and let $\varphi$ be the function on $S$ defined by

$$\varphi := \log \frac{d\mu_S}{dV[\Psi_S]}$$

($dV[\Psi_S]$ may be singular on a subvariety of $S$, also it may be totally singular on $S$).

**Theorem 2.22** *([32, Theorem 5.1]) Let $M$,$S$,$\Psi_S$ be as above. Suppose that $S$ is smooth. Let $d$ be a positive integer such that $d > \alpha m_0$. Then every element of $A^2(S, \mathcal{O}_S(m(K_M + dL)), e^{-(m-1)\varphi} \cdot dV^{-m} \cdot h_L^m \mid_S, dV[\Psi_S])$ extends to an element of*

$$H^0(M, \mathcal{O}_M(m(K_M + dL))).$$

□

As we mentioned as above the smoothness assumption on $S$ is just to make the statement simpler.

Theorem 2.22 follows from Theorem 2.23 below by minor modifications (cf. [32]). The main difference is the fact that the residue volume form $dV[\Psi_S]$ may be singular on $S$. But this does not affect the proof, since in the $L^2$-extension theorem (Theorem 2.17) we do not need to assume that the manifold $M$ is compact. Hence we may remove a suitable subvarieties so that we do not need to consider the pole of $dV[\Psi_S]$ on $S$ (but of course the pole of $dV[\Psi_S]$ affects the $L^2$-conditions).

**Theorem 2.23** *Let $M$ be a projective manifold with a continuous volume form $dV$, let $L$ be a holomorphic line bundle over $M$ with a $C^\infty$-hermitian metric $h_L$ with semipositive curvature $\Theta_{h_L}$, let $S$ be a compact complex submanifold of $M$, let $\Psi_S : M \longrightarrow [-\infty, 0)$ be a continuous function and let $K_M$ be the canonical bundle of $M$.*

1. *$\Psi_S \in \sharp(S) \cap C^\infty(M\backslash S)$ (As for the definition of $\sharp(S)$, see Section 3.2),*
2. *$\Theta_{h_L \cdot e^{-(1+\epsilon)\Psi_S}} \geqq 0$ for every $\epsilon \in [0, \delta]$ for some $\delta > 0$,*
3. *there is a positive line bundle on $M$.*

*Then every element of $H^0(S, \mathcal{O}_S(m(K_M + L)))$ extends to an element of $H^0(M, \mathcal{O}_M(m(K_M + L)))$.* $\square$

For the completeness we shall give a simple proof of Theorem 2.23 (hence also Theorem 2.22) under the additional conditions :

**Conditions**

1. $K_M + L$ is big.
2. Bs$|\ m(K_M + L)\ |$ does not contain $S$ for some $m > 0$.
3. There exists a Zariski open neighbourhood $U$ of the generic point of $S$ in $M$ such that $|\ m(K_M + L)\ |$ gives an embedding of $U$ into a projective space for every sufficiently large $m$.

The reason why we put this condition is that we only need Theorems 2.22 and 2.23 under this condition. More precisely we need to consider the a little bit more general case that $h_L$ is a singular hermitian metric with semipositve curvature current on $M$ and $dV[\Psi]$ is singular on $S$. But as we have already mentioned above the singularity of $dV[\Psi]$ does not change the proof. And the singularity of $h_L$ will be managed in Remark 2.26 below.

Let us begin the proof of Theorem 2.23 under the above additional conditions. Let $M, S, L$ be as in Theorem 2.23. Let $n$ denote the dimension of $M$ and let $k$ denote the dimension of $S$. Let $h_S$ be a canonical AZD ([27]) of $K_M + L\ |_S$. By Kodaira's lemma (cf. [14, Appendix]), there exists an effective $\mathbf{Q}$-divisor $B$ on $M$ such that $K_M + L - B$ is ample. By the above conditions, we may take $B$ such that $\operatorname{Supp} B$ does not contain $S$. In fact by the conditions, we see that for an ample line bundle $H$, $|\ m(K_M + L) - H\ |$ is base point free on the generic point of $S$. Then we may take $B$ to be the $1/m$-times a general member of $|\ m(K_M + L) - H\ |$. We shall assume that $\operatorname{Supp} B$ does not contain $S$.

Let $a$ be a positive integer such that

1. $A := a(K_M + L - B)$ is Cartier,
2. $A\ |_S - K_S$ ia ample and $\mathcal{O}_S(A\ |_S - K_S) \otimes \mathcal{M}_x^{k+1}$ is globally generated for every $x \in S$.

Let $h_M$ be a canonical AZD of $K_M + L$. We shall define a sequence of the hermitian metrics $\{\tilde{h}_m\}(m \geqq 1)$ inductively by :

$$\begin{aligned}\tilde{K}_m &:= K(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}, dV), \\ \tilde{h}_m &:= \frac{1}{\tilde{K}_m},\end{aligned}$$

where $K(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}, dV)$ is the Bergman kernel of $A + m(K_M + L)$ with respect to the singular hermitian metric $dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}$ and the volume form $dV$, i.e.,

$$K(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}, dV) = \sum_j \mid \tilde{\sigma}_j^{(m)} \mid^2,$$

where $\{\tilde{\sigma}_j^{(m)}\}$ is a complete orthonormal basis of $H^0(M, \mathcal{O}_M(A + m(K_M + L)) \otimes \mathcal{I}(\tilde{h}_{m-1}))$ with respect to the inner product

$$(\tilde{\sigma}, \tilde{\sigma}') := \int_M \tilde{\sigma} \cdot \bar{\tilde{\sigma}}' \cdot (dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}) \cdot dV$$

where $\tilde{\sigma}, \tilde{\sigma}' \in H^0(M, \mathcal{O}_M(A + m(K_M + L)) \otimes \mathcal{I}(\tilde{h}_{m-1}))$. We use the similar notation for Bergman kernels hereafter.

Every $\tilde{h}_m$ is a singular hermitian metric on $A + mK_M$ with semipositive curvature current by definition.

**Lemma 2.24** *For every $m \geqq 0$, there exists a positive constant $C_m$ such that*

$$\tilde{h}_m \mid_S \leqq C_m \cdot h_A \mid_S \cdot h_S^m$$

*holds.* $\square$

**Proof.** We shall prove the lemma by induction on $m$. For $m = 0$ the both sides are $\mathcal{O}_S$, hence the inclusion holds. Suppose that the inclusion holds for some $m - 1 \geqq 0$ and a positive constant $C_{m-1}$. Then by the $L^2$-extension theorem, Theorem 2.17 implies that there exists a bounded interpolation operateor :

$$\begin{aligned}I_m : A^2(S, A + m(K_M + L) \mid_S, (dV^{-1} \cdot h_L) \mid_S \cdot \tilde{h}_{m-1} \mid_S, dV[\Psi_S]) \\ \longrightarrow A^2(M, A + m(K_M + L), (dV^{-1} \cdot h_L) \cdot \tilde{h}_{m-1}, dV)\end{aligned}$$

whose operator norm is bounded from above by $C \cdot \delta^{-3/2}$, where $C$ is the positive constant in Theorem 2.17. Hence by the induction assumption, we see that there exists a bounded interpolation operator :

$$\begin{aligned}I'_m : A^2(S, A + m(K_M + L) \mid_S, (dV^{-1} \cdot h_L) \mid_S \cdot (h_A \mid_S \cdot h_S^{m-1}), dV[\Psi_S]) \\ \longrightarrow A^2(M, A + m(K_M + L), (dV^{-1} \cdot h_L) \cdot \tilde{h}_{m-1}, dV)\end{aligned}$$

whose operator norm is bounded from above by $C_{m-1} \cdot C \cdot \delta^{-3/2}$. Let $K(S, A + m(K_M + L) \mid_S, (dV^{-1} \cdot h_L \cdot h_A) \mid_S \cdot h_S^{m-1}, dV[\Psi_S])$ denote the Bergman kernel of $A + m(K_M + L) \mid_S$ with respect to the singular hermitian metric $(dV^{-1} \cdot h_L \cdot h_A) \mid_S \cdot h_S^{m-1}$ and the volume form $dV[\Psi_S]$ (defined as $K(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}, dV)$ above). Then since for every $x \in S$

$$\tilde{K}_m(x) = \sup\{\mid \tilde{\sigma}(x) \mid^2; \tilde{\sigma} \in A^2(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot h_{m-1}, dV), \parallel \tilde{\sigma} \parallel = 1\},$$

and

$$K(S, A + m(K_M + L) \mid_S, (dV^{-1} \cdot h_L \cdot h_A) \mid_S \cdot h_S^{m-1}, dV[\Psi_S])(x)$$

$$= \sup\{\mid \sigma(x) \mid^2; \sigma \in A^2(S, A+m(K_M+L) \mid_S, (dV^{-1}\cdot h_L\cdot h_A) \mid_S \cdot h_S^{m-1}, dV[\Psi_S]), \parallel \sigma \parallel = 1\}$$

hold (cf. [16, p.46, Proposition 1.3.16]), we see that there exists a positive constant $C$ such that

$$\tilde{K}_m \mid_S \geqq (C\cdot\delta^{-3/2})^{-1}\cdot C_{m-1}^{-1}\cdot K(S, A+m(K_M+L) \mid_S, (dV^{-1}\cdot h_L\cdot h_A) \mid_S \cdot h_S^{m-1}, dV[\Psi_S])$$

holds on $S$. Since there exists a positive constant $C_1$ such that

$$dV^{-1} \cdot h_L \leqq C_1 \cdot h_S$$

holds, we see that

$$(\sharp) \qquad \tilde{K}_m \mid_S \geqq (C\cdot\delta^{-3/2})^{-1}\cdot C_{m-1}^{-1}\cdot C_1^{-1}\cdot K(S, A+m(K_M+L) \mid_S, h_A \mid_S \cdot h_S^m, dV[\Psi_S])$$

holds. By the choice of $A$, we see that there exists a positive constant $C_S$ (independent of $m$, although this fact is not used in the proof) such that

$$(\flat) \qquad K(S, A + m(K_M + L) \mid_S, h_A \mid_S \cdot h_S^m, dV[\Psi_S]) \geqq C_S \cdot (h_A \mid_S \cdot h_S^m)^{-1}$$

holds. This can be verified as follows. Since $A \mid_S - K_S$ is ample, we see that there exists a $C^\infty$-hermitian metric $h_{A/S}$ on $A \mid_S$ such that the hermitian metric $dV[\Psi_S] \cdot h_{A/S}$ on $A \mid_S - K_S$ has strictly positive curvature everywhere on $S$.

Let $x$ be a point on $M$ and $\{\sigma_{A,q}\}$ a basis of $H^0(S, \mathcal{O}_S(A \mid_S - K_S) \otimes \mathcal{M}_x^{k+1})$. Then in Theorem 2.17 (see also Lemma 2.18), taking $\Psi$ to be

$$\Psi_x := \frac{k}{k+1} \log \sum_q dV[\Psi_S] \cdot h_{A/S}(\sigma_{A,q}, \sigma_{A,q}),$$

and $(E, h_E)$ to be

$$(A \mid_S - K_S + m(K_M + L) \mid_S, dV[\Psi_S] \cdot h_{A/S} \cdot h_S^m),$$

by Theorem 2.17 and Lemma 2.18, we have a bounded interpolation operator :

$$I_{m,x} : A^2(x, A+m(K_M+L) \mid_x, h_{A/S}\cdot h_S^m \mid_x, \delta_x) \longrightarrow A^2(S, A+m(K_M+L), h_{A/S}\cdot h_S^m, dV[\Psi_S]),$$

where $\delta_x$ is the Dirac measure at $x$. We note that by the definition of $\Psi_x$ and the fact that $\mathcal{O}_S(A \mid_S - K_S) \otimes \mathcal{M}_x^{k+1}$ is globally generated, $\log \Psi_x$ has singularity only at $x$ and the operator norm of the $I_{m,x}$ is less than or equal to $C \cdot k^{3/2}$ by Theorem 2.17, where $C$ is the positive constant in Theorem 2.17. Hence we see that

$$K(S, A + m(K_M + L) \mid_S, h_{A/S} \cdot h_S^m, dV[\Psi_S]) \geqq C^{-1} \cdot k^{-3/2} \cdot (h_{A/S} \cdot h_S^m)^{-1}$$

holds by the basic property of Bergman kernels (cf.[16, p.46, Proposition 1.3.16]). We note that $h_{A/S}$ is quasi-isometric to $h_A \mid_S$, i.e., there exists a positive constant $C_{A,S} > 1$ such that

$$C_{A,S}^{-1} \cdot h_{A/S} \leqq h_A \mid_S \leqq C_{A,S} \cdot h_{A/S}$$

holds on $S$. Then this implies that

$$K(S, A+m(K_M+L)\mid_S, h_A\mid_S \cdot h_S^m, dV[\Psi_S]) \geqq C_{A,S}^{-1}\cdot C^{-1}\cdot k^{-3/2}\cdot(h_A\mid_S \cdot h_S^m)^{-1}$$

holds on $S$. This is the desired estimate ($\flat$) with $C_S = C_{A,S}^{-1}\cdot C^{-1}\cdot k^{-3/2}$.

Combining ($\sharp$) and ($\flat$), we see that

$$\tilde{K}_m\mid_S \geqq (C\cdot\delta^{-3/2})^{-1}\cdot C_{m-1}^{-1}\cdot C_1^{-1}\cdot C_S\cdot(h_A\mid_S\cdot h_S^m)^{-1}$$

holds on $S$. Then by the definition of $\tilde{h}_m$, we see that

$$\tilde{h}_m\mid_S \leqq (C\cdot\delta^{-3/2})\cdot C_1\cdot C_S^{-1}\cdot C_{m-1}\cdot h_A\mid_S\cdot h_S^m$$

holds. Hence we complete the proof of Lemma 2.24 by induction on $m$. □

By the definition of $A$, we may consider the metric $h_A$ as a singular hermitian metric $\hat{h}_A$ on $a(K_M+L)$. Also we may consider $\tilde{h}_m$ as a singular hermitian metric on $\hat{h}_m$ on $(a+m)(K_M+L)$. Then by Lemma 2.24, we have the following lemma.

**Lemma 2.25** *For every $m \geqq 0$, there exist a positive constant $C'_m$ depending on $m$ and a positive constant $C$ independent of $m$ such that*

$$h_M^{a+m}\mid_S \leqq C'_m\cdot\hat{h}_m\mid_S \leqq C^{m+1}\hat{h}_A\mid_S\cdot h_S^m$$

*hold.* □

By Lemma 2.25, we see that

$$h_M \leqq (C'_m)^{\frac{1}{a+m}}\cdot\hat{h}_A\mid_S^{\frac{1}{a+m}}\cdot h_S^{\frac{m}{a+m}}$$

holds.

Let us fix an arbitrary nonnegative integer $\ell$. Then since $h_S$ is an AZD of $K_M+L\mid_S$,

$$\{\mathcal{I}(\hat{h}_A\mid_S^{\frac{\ell}{a+m}}\cdot h_S^{\frac{m}{a+m}\ell})\}_{m=1}^{\infty}$$

is an increasing sequence of ideal sheaves on $S$ contained in $\mathcal{I}(h_S^\ell)$. Let $\phi,\rho$ be a weight functions of $h_S^{\frac{m}{a+m}\ell}$ and $\hat{h}_A\mid_S^{\frac{\ell}{a+m}}$ with respect to (the powers of) $dV^{-1}\cdot h_L\mid_S$ respectively. By Hölder's inequality we see that for a holomorphic function $f$ on an open set $V$ in $S$,

$$\int_V e^{-\phi}\cdot e^{-\rho}\cdot\mid f\mid^2 dV[\Psi_S] \leqq (\int_V e^{-p\phi}\cdot\mid f\mid^2 dV[\Psi_S])^{\frac{1}{p}}\cdot(\int_V e^{-q\rho}\cdot\mid f\mid^2 dV[\Psi_S])^{\frac{1}{q}}$$

holds, where

$$p := (1+\frac{1}{\ell})(1+\frac{m}{a}), q = \frac{p}{p-1}.$$

Since

$$e^{-p\phi}\cdot(dV^{-1}\cdot h_L\mid_S)^{\ell+1} = h_S^{\ell+1}$$

holds, this implies that there exists a positive integer $m_\ell$ depending on $\ell$ such that

$$\mathcal{I}(\hat{h}_A \mid_S^{\frac{\ell}{a+m_\ell}} \cdot h_S^{\frac{m_\ell}{a+m_\ell}\ell}) \supseteq \mathcal{I}(h_S^{\ell+1})$$

holds. Hence we see that

$$\mathcal{I}(h_M \mid_S^{\ell}) \supseteq \mathcal{I}(h_S^{\ell+1})$$

holds on $S$. We note that since $h_S$ is an AZD of $(K_M + L) \mid_S$,

$$A^2(S, (\ell+1)(K_M+L) \mid_S, h_S^{\ell+1}, dV[\Psi_S]) \simeq A^2(S, (\ell+1)(K_M+L) \mid_S, dV^{-1} \cdot h_L \mid_S \cdot h_S^{\ell}, dV[\Psi_S])$$

holds. Using this equality, by Theorem 2.17 (and Lemma 2.18) in Section 2.3, we see that every element of

$$A^2(S, (\ell+1)(K_M+L) \mid_S, dV^{-1} \cdot h_L \mid_S \cdot h_S^{\ell}, dV[\Psi_S])$$

can be extended to an element of

$$A^2(M, (\ell+1)(K_M+L), dV^{-1} \cdot h_L \cdot h_M^{\ell}, dV).$$

Since $\ell$ is an arbitrary nonnegative integer, we complete the proof of Theorem 2.23. $\square$.

**Remark 2.26** *The above proof also works for the case that* $(L, h_L)$ *is a singular hermitian line bundle with semipositive curvature current, if we assume the following conditions :*

1. $(K_M + L, dV^{-1} \cdot h_L)$ *is big.*
2. $Bs| \, m(K_M + L, dV^{-1} \cdot h_L) \mid_\infty$ *does not contain* $S$ *for some* $m > 0$.
3. *There exists a Zariski open neighbourhood* $U$ *of the generic point of* $S$ *in* $M$ *such that* $\mid m(K_M + L, dV^{-1} \cdot h_L) \mid_\infty$ *gives an embedding of* $U$ *into a projective space for every sufficiently large* $m$.

*Here* $\mid m(K_M+L, dV^{-1} \cdot h_L) \mid_\infty$ *denotes the linear system* $\mid H^0(M, \mathcal{O}_M(m(K_M + L)) \otimes \mathcal{I}_\infty(h_L^m)) \mid$. *In this case we need to take an AZD* $h_S$ *of the singular hermitian line bundle* $(K_M + L, dV^{-1} \cdot h_L) \mid_S$. *Noting Remarks 2.14 and 2.16, by Corollary 2.15 there exists an AZD* $h_S$ *of* $(K_M + L, dV^{-1} \cdot h_L) \mid_S$. $\square$

**Remark 2.27** *The full proofs of Theorems 2.22 and 2.23 can be obtained similar line as the above proof. But they require more detailed estimates. The proof presented here is somewhat similar to the argument in [24].* $\square$

## 2.7 Positivity result

The following positivity theorem is the key to the proof of Theorems 1.1 and 1.2.

**Theorem 2.28** *([11, p.894,Theorem 2]) Let* $f : X \longrightarrow B$ *be a surjective morphism of smooth projective varieties with connected fibers. Let* $P = \sum P_j$ *and* $Q = \sum_\ell Q_\ell$ *be normal crossing divisors on* $X$ *and* $B$ *respectively, such that* $f^{-1}(Q) \subset P$ *and* $f$ *is smooth over* $B \backslash Q$. *Let* $D = \sum d_j P_j$ *be a* $\mathbf{Q}$*-divisor on* $X$, *where* $d_j$ *may be positive, zero or negative, which satisfies the following conditions :*

1. *$D = D^h + D^v$ such that $f : Supp(D^h) \to B$ is surjective and smooth over $B \backslash Q$, and $f(Supp(D^v)) \subset Q$. An irreducible component of $D^h$(resp. $D^v$) is called horizontal (resp. vertical).*

2. *$d_j < 1$ for all $j$.*

3. *The natural homomorphism $\mathcal{O}_B \to f_*\mathcal{O}_X(\lceil -D \rceil)$ is surjective at the generic point of $B$.*

4. *$K_X + D \sim_{\mathbf{Q}} f^*(K_B + L)$ for some $\mathbf{Q}$-divisor $L$ on $B$.*

*Let*

$$\begin{aligned} f^*Q_\ell &= \sum_j w_{\ell j} P_j \\ \bar{d}_j &:= \frac{d_j + w_{\ell j} - 1}{w_{\ell j}} \quad \text{if} \ \ f(P_j) = Q_\ell \\ \delta_\ell &:= \max\{\bar{d}_j; f(P_j) = Q_\ell\} \\ \Delta &:= \sum_\ell \delta_\ell Q_\ell \\ M &:= L - \Delta. \end{aligned}$$

*Then $M$ is nef.* □

Here the meaning of the divisor $\Delta$ may be difficult to understand. So I would like to give an geometric interpretation of $\Delta$. Let $X, P, Q, D, B, \Delta$ be as above. Let $dV$ be a $C^\infty$-volume form on $X$. Let $\sigma_j$ be a global section of $\mathcal{O}_X(P_j)$ with divisor $P_j$. Let $\parallel \sigma_j \parallel$ denote the hermitian norm of $\sigma_j$ with respect to a $C^\infty$-hermitian metric on $\mathcal{O}_X(P_j)$ respectively. Let us consider the singular volume form

$$\Omega := \frac{dV}{\prod_j \parallel \sigma_j \parallel^{2d_j}}$$

on $X$. Then by taking the fiber integral of $\Omega$ with respect to $f : X \longrightarrow B$, we obtain a singular volume form $\int_{X/B} \Omega$ on $B$, where the fiber integral $\int_{X/B} \Omega$ is defined by the property that for any open set $U$ in $B$,

$$\int_U (\int_{X/B} \Omega) = \int_{f^{-1}(U)} \Omega$$

holds. We note that the condition 2 in Theorem 2.28 assures that $\int_{X/B} \Omega$ is continuous on a nonempty Zariski open subset of $B$. Also by the condition 4 in Theorem 2.28, we see that $K_X + D$ is numerically $f$-trivial and $(\int_{X/B} \Omega)^{-a}$ is a $C^0$-hermitian metric on a line bundle $a(K_B + \Delta)$, where $a$ is a positive integer such that $a\Delta$ is Cartier. Thus the divisor $\Delta$ corresponds exactly to singularities (poles and degenerations) of the singular volume form $\int_{X/B} \Omega$ on $B$.

# 3 Proofs of Theorems 1.1 and 1.2

In this section we shall prove Theorems 1.1 and 1.2 simultaneously. The proof is almost parallel to the one assuming MMP ([34]), if we replace the minimal model by an AZD (analytic Zariski decomposition) of the canonical line bundle.

### 3.1 Construction of a filtration

Let $X$ be a smooth projective $n$-fold of general type. Let $h$ be an AZD of $K_X$ constructed as in Section 2.2. We may assume that $h$ is lowersemicontinuous by Theorem **??**. This is a technical assumption so that a local potential of the curvature current of $h$ is plurisubharmonic. This is used to restrict $h$ to a subvariety of $X$ (if we only assume that the local potential is only locally integrable, the restriction is not well defined). We set

$$X^\circ = \{x \in X \mid x \not\in \mathrm{Bs} \mid mK_X \mid \text{ and } \Phi_{|mK_X|} \text{ is a biholomorphism}$$
$$\text{on a neighbourhood of } x \text{ for some } m \geqq 1\}.$$

We set

$$\mu_0 := (K_X, h)^n = \mu(X, (K_X, h)) = \mu(X, K_X).$$

For the notations $(K_X, h)^n, \mu(X, (K_X, h))$ and $\mu(X, K_X)$ see Definitions 2.20 and 2.19. The last equality holds, since $h$ is an AZD of $K_X$. We note that for every $x \in X^\circ$, $\mathcal{I}(h^m)_x \simeq \mathcal{O}_{X,x}$ holds for every $m \geqq 0$ (cf. [27] or [6, Theorem 1.5]).

Let $x, x'$ be distinct points on $X^\circ$. In this subsection we shall construct a filtration

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = x \text{ or } x'$$

of $X$ by a strictly decreasing sequence of subvarieties $\{X_i\}_{i=0}^{r+1}$ for some $r$ (depending $x$ and $x'$) and invariants :

$$\alpha_0, \alpha_1, \ldots, \alpha_r,$$

$$n =: n_0 > n_1 > \cdots > n_r \quad (n_i = \dim X_i, i = 0, \cdots, r)$$

and

$$\mu_0, \mu_1, \ldots, \mu_r \quad (\mu_i = (K_X, h)^{n_i} \cdot X_i, i = 0, \cdots, r)$$

(cf. Definition 2.20) with the estimates

$$\alpha_i \leqq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + \delta \quad (0 \leqq i \leqq r),$$

where $\delta$ is a fixed positive number less than $1/n$.

**Lemma 3.1** *We set*

$$\mathcal{M}_{x,x'} = \mathcal{M}_x \cdot \mathcal{M}_{x'},$$

*where $\mathcal{M}_x, \mathcal{M}_{x'}$ denote the maximal ideal sheaves of the points $x, x'$ respectively. Let $\varepsilon$ be positive number strictly less than 1. Then*

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m) \cdot \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}} \rceil}) \neq 0$$

*holds for every sufficiently large $m$.* $\square$

**Proof**. First we note that since $x, x' \in X^\circ$, $h$ is bounded from above at $x$ and $x'$ by the construction of $h$ (cf. Theorem 2.11). In particular $\mathcal{I}(h^m)_x = \mathcal{O}_{X,x}$ and $\mathcal{I}(h^m)_{x'} = \mathcal{O}_{X,x'}$ hold for every $m \geqq 0$. Let us consider the exact sequence:

$$0 \to H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m) \cdot \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}} \rceil}) \to H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m)) \to$$

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m) \otimes \mathcal{O}_X/\mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}} \rceil}).$$

We note that

$$n! \cdot \overline{\lim}_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m)) = \mu_0$$

holds by the definition of $\mu_0$.

On the other hand, we see that

$$n! \cdot \overline{\lim}_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m) \otimes \mathcal{O}_X/\mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}} \rceil}) = \mu_0(1-\varepsilon)^n < \mu_0$$

hold, since $\mathcal{I}(h^m)_x = \mathcal{O}_{X,x}$ and $\mathcal{I}(h^m)_{x'} = \mathcal{O}_{X,x'}$ hold for every $m \geqq 0$.

By the above inequalities and the exact sequence, we complete the proof of Lemma 3.1. $\square$

Let $\varepsilon$ be a positive number less than 1 as in Lemma 3.1. Let us take a sufficiently large positive integer $m_0$ such that

$$H^0(X, \mathcal{O}_X(m_0K_X) \otimes \mathcal{I}(h^{m_0}) \cdot \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m_0}{\sqrt[n]{2}} \rceil}) \neq 0$$

as in Lemma 3.1 and let

$$\sigma_0 \in H^0(X, \mathcal{O}_X(m_0K_X) \otimes \mathcal{I}(h^{m_0}) \cdot \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m_0}{\sqrt[n]{2}} \rceil})$$

be a general nonzero element. We set

$$D_0 := \frac{1}{m_0}(\sigma_0)$$

and

$$h_0 := \frac{1}{\mid \sigma_0 \mid^{2/m_0}}$$

(see Example 2.2 in Section 2.1 for the meaning of $1/\mid \sigma_0 \mid^{2/m_0}$ ). We define the positive number $\alpha_0$ by

$$\alpha_0 := \inf\{\alpha > 0 \mid (X, \alpha D_0) \text{ is KLT at neither } x \text{ nor } x'\},$$

where KLT is short for of Kawamata log terminal (cf. Definition 2.4).

Let $\mu : Y \longrightarrow X$ be a log resolution of $(X, D)$ and for $\alpha > 0$ let

$$K_Y + \mu_*^{-1}(\alpha D) = \mu^*(K_X + \alpha D) + F(\alpha),$$

where $F(\alpha)$ denotes the discrepancy depending on $\alpha$. Then $\alpha_0$ is the infimum of $\alpha$ such that the discrepancy $F(\alpha)$ has a component whose coefficient is less than or equal to $-1$. Hence by the construction $\alpha_0$ is a rational number.

Since $(\sum_{i=1}^{n} \mid z_i \mid^2)^{-n}$ is not locally integrable around $O \in \mathbf{C}^n$, by the definition of $D_0$, we see that

$$\alpha_0 \leqq \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}(1-\varepsilon)}$$

holds. About the relation between the KLT conditions and the multiplier ideal sheaves, please see Section 2.1.

Let $\delta$ be the fixed positive number as above and let us make $\varepsilon > 0$ sufficiently small so that

$$\alpha_0 \leqq \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}} + \delta$$

holds. Then one of the following two cases occurs.

**Case** 1: For every sufficiently small positive number $\lambda$, $(X, (\alpha_0 - \lambda)D_0)$ is KLT at both $x$ and $x'$.
**Case** 2: For every sufficiently small positive number $\lambda$, $(X, (\alpha_0 - \lambda)D_0)$ is KLT at exactly one of $x$ or $x'$, say $x$.

We define the next stratum $X_1$ by

$X_1 :=$ the minimal center of log canonical singularities of $(X, \alpha_0 D_0)$ at $x$ (cf. Section 2.1).

If $X_1$ is a point, we stop the construction of the filtration. Suppose that $X_1$ is not a point.

Case 1 divides into the following two cases.

**Case** 1.1: $X_1$ passes through both $x$ and $x'$,
**Case** 1.2: $X_1$ passes through only one of $x$ and $x'$, say $x$.

First we shall consider Case1.1. We define the positive number $\mu_1$ by

$$\mu_1 := \mu(X_1, (K_X, h) \mid_{X_1}).$$

Then since $x, x' \in X^\circ$, $\mu_1$ is positive.

For the later purpose, we shall modify $h_0$ so that $X_1$ is the only center of log canonical singularities of $(X, \alpha_0 D_0)$ at $x$. Let us take an effective $\mathbf{Q}$-divisor $G$ such that $K_X - G$ is ample by Kodaira's lemma (cf.[14, Appendix]). By the definition of $X^\circ$, we may assume that the support of $G$ contains neither $x$ nor $x'$. In fact this can be verified as follows. Let $H$ be an arbitrary ample divisor on $X$. Then by the definition of $X^\circ$, $\mid bK_X - H \mid$ is base point free at $x$ and $x'$ for every sufficiently large $b$. Fix such a $b$ and take a member $G'$ of $\mid bK_X - H \mid$ which contains neither $x$ nor $x'$. Then we may take $G$ to be $b^{-1}G'$.

Let $a$ be a positive integer such that $A := a(K_X - G)$ is a very ample Cartier divisor such that $\mathcal{O}_X(A) \otimes \mathcal{I}_{X_1}$ is globally generated. Let $\rho_1, \ldots, \rho_e \in H^0(X, \mathcal{O}_X(A) \otimes \mathcal{I}_{X_1})$ be a set of generators of $\mathcal{O}_X(A) \otimes \mathcal{I}_{X_1}$ on $X$. Then if we replace $h_0$ by

$$\frac{1}{(\mid \sigma_0 \mid^2 (\sum_{i=1}^{e} \mid \rho_i \mid^2))^{\frac{1}{m_0 + a}}},$$

it has the desired property. If we take $m_0$ very large (in comparison with $a$), we can make the new $\alpha_0$ arbitrary close to the original $\alpha_0$. Hence we may assume that the estimate

$$\alpha_0 \leqq \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}} + \delta$$

still holds after the modification. Let us set

$$n_1 := \dim X_1.$$

The proof of the following lemma is identical to that of Lemma 3.1.

**Lemma 3.2** *Let $\varepsilon'$ be a positive rational number less than* 1. *Let $x_1, x_2$ be distinct regular points of $X_1 \cap X^\circ$. Then for every sufficiently large positive integer $m$*

$$H^0(X_1, \mathcal{O}_{X_1}(mK_X) \otimes \mathcal{I}(h^m\mid_{X_1}) \cdot \mathcal{M}_{x_1,x_2}^{\lceil n\sqrt[n]{\mu_1}(1-\varepsilon')\frac{m}{n\sqrt[n]{2}}\rceil}) \neq 0$$

*holds.* $\square$

Let $x_1, x_2$ be two distinct regular points of $X_1 \cap X^\circ$. Let $m_1$ be a positive integer such that

$$H^0(X_1, \mathcal{O}_{X_1}(m_1K_X) \otimes \mathcal{I}(h^{m_1}\mid_{X_1}) \cdot \mathcal{M}_{x_1,x_2}^{\lceil n\sqrt[n]{\mu_1}(1-\varepsilon')\frac{m}{n\sqrt[n]{2}}\rceil}) \neq 0$$

as in Lemma 3.2 and let

$$\sigma'_{1,x_1,x_2} \in H^0(X_1, \mathcal{O}_{X_1}(m_1K_X) \otimes \mathcal{I}(h^{m_1}\mid_{X_1}) \cdot \mathcal{M}_{x_1,x_2}^{\lceil n\sqrt[n]{\mu_1}(1-\varepsilon')\frac{m}{n\sqrt[n]{2}}\rceil})$$

be a nonzero element.

We shall extend the singular hermitian metric $1/\mid \sigma'_{1,x_1,x_2}\mid^{2/m_1}$ of $K_X\mid_{X_1}$ to a singular hermitian metric on $K_X$ with semipositive curvature current after a modification.

As before by Kodaira's lemma ([14, Appendix]) there is an effective **Q**-divisor $G$ such that $K_X - G$ is ample. By the definition of $X^\circ$, we may assume that the support of $G$ contains neither $x$ nor $x'$ as before. Let $\ell_1$ be a sufficiently large positive integer which will be specified later such that

$$L_1 := \ell_1(K_X - G)$$

is Cartier. Let $h_{L_1}$ be a $C^\infty$-hermitian metric on $L_1$ with strictly positive curvature. Let $\tau$ be a nonzero section in $H^0(X, \mathcal{O}_X(L_1))$. We set

$$\Psi := \alpha_0 \cdot \log\frac{h}{h_0}.$$

Let $dV$ be a $C^\infty$-volume form on $X$. We note that the residue volume form $dV[\Psi]$ on $X_1$ may have poles along some proper subvarieties in $X_1$. By taking $\ell_1$ sufficiently large and taking $\tau$ properly, we may assume that $h_{L_1}(\tau,\tau)\cdot dV[\Psi]$ is nonsingular on $X_1$ in the sense that the pullback of it to a nonsingular model of $X_1$ is a bounded form. Then by applying Lemma 2.18 for $S = X_1$, $\Psi = \alpha_0\log(h/h_0)$,

$$(E, h_E) = ((\lceil 1+\alpha_0\rceil)K_X, h^{\lceil 1+\alpha_0\rceil}),$$

and

$$(L, h_L) = ((m_1 - \lceil \alpha_0 \rceil - 2)K_X + L_1, h^{(m_1 - \lceil \alpha_0 \rceil - 2)} \otimes h_{L_1}),$$

we see that

$$\sigma'_{1,x_1,x_2} \otimes \tau \in H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X + L_1) \otimes \mathcal{I}(h^{m_1} \mid_{X_1}) \cdot \mathcal{M}_{x_1,x_2}^{\lceil \sqrt[n]{\mu_1}(1-\varepsilon')\frac{m_1}{\sqrt[n]{2}} \rceil})$$

extends to a section

$$\sigma_{1,x_1,x_2} \in H^0(X, \mathcal{O}_X((m_1 + \ell_1)K_X)).$$

We note that even though $dV[\Psi]$ may be singular on $X_1$, we may apply Lemma 2.18, because there exists a proper Zariski closed subset $B$ of $X$ such that the restriction of $dV[\Psi]$ to $(X - B) \cap X_1$ is smooth. Of course the singularity of $dV[\Psi]$ affects the $L^2$-condition. But this has already been managed by the boundedness of $h_{L_1}(\tau, \tau) \cdot dV[\Psi]$.

Taking $\ell_1$ sufficiently large, we may and do assume that there exists a neighbourhood $U_{x_1,x_2}$ of $\{x_1, x_2\}$ such that the divisor $(\sigma_{1,x_1.x_2})$ is smooth on $U_{x_1,x_2} \backslash X_1$. This can be verified as follows. Let us take $\ell_1$ sufficiently large so that $\mathcal{O}_X(L_1) \otimes \mathcal{M}_y^{n+1}$ is globally generated for every $y \in X$. Let us fix $y$ and let $\{\xi_1, \cdots, \xi_N\}$ be a set of basis of $H^0(X, \mathcal{O}_X(L_1) \otimes \mathcal{M}_y^{n+1})$. Then

$$h_{L_1,y} := h_{L_1}^{\frac{1}{n+1}} \cdot \left(\frac{1}{\sum_{j=1}^{N} \mid \xi_j \mid^2}\right)^{\frac{n}{n+1}}$$

is a singular hermitian metric of $L_1$ with strictly positive curvature current. Since $\mathcal{O}_X(L_1) \otimes \mathcal{M}_y^{n+1}$ is globally generated, we see that $\mathcal{O}_X / \mathcal{I}(h_{L_1,y})$ has isolated support at $y$. By Nadel's vanishing theorem [20, p.561], this implies that for every $y \in X^\circ \backslash X_1$,

$$H^1(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}(h_0^{\alpha_0} \cdot h^{m-1-\alpha_0}) \otimes \mathcal{M}_y) = 0$$

holds. Hence for every $y \in X^\circ \backslash X_1$, we may modify the $L^2$-extension of $\sigma'_{1,x_1,x_2} \otimes \tau$ so that the extension has any prescribed value at $y$, if we take $\ell_1$ is sufficiently large. We may take $\ell_1$ to be independent of $y \in X^\circ \backslash X_1$. Then by Bertini's theorem we may find a neighbourhood $U_{x_1,x_2}$ of $\{x_1, x_2\}$ such that the divisor $(\sigma_{1,x_1.x_2})$ is smooth on $U_{x_1,x_2} \backslash X_1$.

We set

$$D_1(x_1, x_2) := \frac{1}{m_1 + \ell_1}(\sigma_{1,x_1,x_2}).$$

Let $X_{1,\mathrm{reg}}$ denote the set of regular points on $X_1$. We may construct the divisors $\{D_1(x_1, x_2)\}$ as an algebraic family over $(X_{1,\mathrm{reg}} \times X_{1,\mathrm{reg}}) \setminus \Delta_{X_1}$ where $\Delta_{X_1}$ denotes the diagonal of $X_1 \times X_1$. This construction is possible, since we may take $L_1$ idependent of $x_1, x_2 \in X_{1,\mathrm{reg}}$. Letting $x_1$ and $x_2$ tend to $x$ and $x'$ respectively, we obtain a $\mathbf{Q}$-divisor $D_1$ on $X$ which is $(m_1 + \ell_1)^{-1}$-times a divisor of a global holomorphic section

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1 + \ell_1)K_X)).$$

By the construction, we may and do assume that there exists a neighbourhood $U_{x,x'}$ of $\{x, x'\}$ such that $(\sigma_1)$ is smooth on $U_{x,x'} \backslash X_1$.

Let $\varepsilon_0$ be a positive rational number with $\varepsilon_0 < \alpha_0$. And we define the positive numbers $\alpha_1(x_1, x_2)$ and $\alpha_1$ by

$$\alpha_1(x_1,x_2) := \inf\{\alpha > 0 \mid \ (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1(x_1,x_2) \text{ is KLT at neither } x_1 \text{ nor } x_2\}$$

and

$$\alpha_1 := \inf\{\alpha > 0 \mid \ (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1 \text{ is KLT at neither } x \text{ nor } x'\}$$

respectively. For every positive number $\lambda$, $(\alpha_0 - \varepsilon_0)D_0 + (\alpha_1 - \lambda)D_1$ is KLT at $x$ or $x'$, say $x$. Then we shall define the proper subvariety $X_2$ of $X_1$ by

$X_2 :=$ the minimal center of log canonical singularities of $(X, (\alpha_0 - \varepsilon_0)D_0 + \alpha_1 D_1)$ at $x$.

We shall estimate $\alpha_1$. We note that $m_1$ is independent of $\ell_1$ in the extension of $\sigma'_{1,x_1,x_2} \otimes \tau$.

**Lemma 3.3** *Let $\delta$ be the fixed positive number as above, then we may assume that*

$$\alpha_1 \leqq \frac{n_1 \sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}} + \delta$$

*holds, if we take $\varepsilon'$, $\ell_1/m_1$ and $\varepsilon_0$ sufficiently small.* $\square$

To prove Lemma 3.3, we need the following elementary lemma.

**Lemma 3.4** *([30, p.12, Lemma 6]) Let $a, b$ be positive numbers. Then*

$$\int_0^1 \frac{r_2^{2n_1-1}}{(r_1^2 + r_2^{2a})^b} dr_2 = r_1^{\frac{2n_1}{a} - 2b} \int_0^{r_1^{-2a}} \frac{r_3^{2n_1-1}}{(1 + r_3^{2a})^b} dr_3$$

*holds, where*

$$r_3 = r_2 / r_1^{1/a}.$$

$\square$

**Proof of Lemma 3.3.** First suppose that $x, x'$ are **nonsingular points** on $X_1$. Then we may set $x_1 = x, x_2 = x'$, i.e., we do not need the limiting process to define the divisor $D_1$. Let $(z_1, \ldots, z_n)$ be a local coordinate system on a neighbourhood $U$ of $x$ in $X$ such that

$$U \cap X_1 = \{q \in U \mid z_{n_1+1}(q) = \cdots = z_n(q) = 0\}.$$

We set $r_1 = (\sum_{i=n_1+1}^{n} \mid z_1 \mid^2)^{1/2}$ and $r_2 = (\sum_{i=1}^{n_1} \mid z_i \mid^2)^{1/2}$. Fix an arbitrary $C^\infty$-hermitian metric $h_K$ on $K_X$. Then there exists a positive constant $C$ such that

$$(\star) \qquad \parallel \sigma_1 \parallel^2 \leqq C(r_1^2 + r_2^{2\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')\frac{m_1}{\sqrt[n_1]{2}} \rceil})$$

holds on a neighbourhood of $x$, where $\parallel \ \parallel$ denotes the norm with respect to $h_K^{m_1+\ell_1}$. We note that there exists a positive integer $M$ such that

$$\parallel \sigma_1 \parallel^{-2} = O(r_1^{-M})$$

holds on a neighbourhood of the generic point of $U \cap X_1$, where $\parallel \ \parallel$ denotes the norm with respect to $h_K^{m_0}$. Let us apply Lemma 3.4 by taking

$$a = \lceil \sqrt[n_1]{\mu_1}(1 - \varepsilon')\frac{m_1}{\sqrt[n_1]{2}}\rceil .$$

Then by Lemma 3.4 and the estimate $(\star)$, we see that for every

$$b > \frac{n_1}{\lceil \sqrt[n_1]{\mu_1}(1 - \varepsilon')\frac{m_1}{\sqrt[n_1]{2}}\rceil}.$$

$\parallel \sigma_1 \parallel$ produces a singularity greater than equal to $r_1^{\frac{2n_1}{a} - b}$, if we average the singularity in terms of the volume form in $z_1, \cdots, z_{n_1}$ direction. Hence by Proposition 2.5, we have the inequality:

$$\alpha_1 \leqq (\frac{m_1 + \ell_1}{m_1})\frac{n_1 \sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}(1 - \varepsilon')} + m_1\varepsilon_0.$$

Taking $\varepsilon'$, $\ell_1/m_1$ and $\varepsilon_0$ sufficiently small, we obtain that

$$\alpha_1 \leqq \frac{n_1 \sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}} + \delta$$

holds. $\square$

If $x$ or $x'$ is a singular point on $X_1$, we need the following lemma.

**Lemma 3.5** *Let $\varphi$ be a plurisubharmonic function on $\Delta^n \times \Delta$. Let $\varphi_t (t \in \Delta)$ be the restriction of $\varphi$ on $\Delta^n \times \{t\}$. Assume that $e^{-\varphi_t}$ does not belong to $L^1_{loc}(\Delta^n, O)$ for any $t \in \Delta^*$.*

*Then $e^{-\varphi_0}$ is not locally integrable at $O \in \Delta^n$.* $\square$

Lemma 3.5 is an immediate consequence of the $L^2$-extension theorem [22, p.20, Theorem]. Using Lemma 3.5 and Lemma 3.4, letting $x_1 \to x$ and $x_2 \to x'$, we see that

$$\alpha_1 \leqq \liminf_{x_1 \to x, x_2 \to x'} \alpha_1(x_1, x_2)$$

holds.

Next we consider Case 2. The remaining case Case 1.2 will be considered later. In Case 2, for every sufficiently small positive number $\lambda$, $(X, (\alpha_0 - \lambda)D_0)$ is KLT at $x$ and not KLT at $x'$. In Case 1.2, instead of Lemma 3.2, we use the following simpler lemma. We define $X_1$ as before.

In this case, instead of Lemma 3.2, we use the following simpler lemma.

**Lemma 3.6** *Let $\varepsilon'$ be a positive number less than 1 and let $x_1$ be a smooth point on $X_1$. Then for a sufficiently large $m > 1$,*

$$H^0(X_1, \mathcal{O}_{X_1}(mK_X) \otimes \mathcal{I}(h^m \mid_{X_1}) \cdot \mathcal{M}_{x_1}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon')m\rceil}) \neq 0$$

*holds.* $\square$

Let us take a general nonzero element $\sigma'_{1,x_1}$ in

$$H^0(X_1, \mathcal{O}_{X_1}(m_1K_X) \otimes \mathcal{I}(h^{m_1} \mid_{X_1}) \cdot \mathcal{M}_{x_1}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon)m_1 \rceil}),$$

for a sufficiently large $m_1$. Using Lemma 3.6, let $\ell_1$ be as in Lemma 3.3 and let $\tau$ be a general nonzero section in $H^0(X, \mathcal{O}_X(L_1))$, where $L_1$ is the line bundle as in Lemma 3.3. By Lemma 3.3, we may extend $\sigma_{1,x'_1} \otimes \tau$ to a section

$$\sigma_{1,x_1} \in H^0(X, \mathcal{O}_X((m_1 + \ell_1)K_X)).$$

As in Case 1.1, taking $\ell_1$ sufficiently large, we may assume that there exists a neighbourhood $U_{x_1}$ of $x_1$ such that $(\sigma_{1,x_1})$ is smooth on a $U_{x_1}\backslash X_1$. We set

$$D_1(x_1) = \frac{1}{m_1 + \ell_1}(\sigma_{1,x_1}).$$

Let $X_{1,\mathrm{reg}}$ denote the regular locus of $X_1$. We may construct the divisors $\{D_1(x_1)\}$ as an algebraic family over $X_{1,\mathrm{reg}}$. Letting $x_1$ tend to $x$, we obtain a $\mathbf{Q}$-divisor $D_1$ on $X$ which is $(m_1 + \ell_1)^{-1}$-times a divisor of a global holomorphic section

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1 + \ell_1)K_X)).$$

By the construction, we may and do assume that there exists a neighbourhood $U_x$ of $x$ such that $(\sigma_1)$ is smooth on $U_x\backslash X_1$. Let $\varepsilon_0$ be a sufficiently small positive rational number with $\varepsilon_0 < \alpha_0$ such that $(\alpha_0 - \varepsilon_0)D_0$ is not KLT at $x'$ (this is possible because we are considering Case 2).

And we define the positive numbers $\alpha_1(x_1)$ and $\alpha_1$ by

$$\alpha_1(x_1) := \inf\{\alpha > 0 \mid \ (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1(x_1) \text{ is not KLT at } x_1\}.$$

and

$$\alpha_1 := \inf\{\alpha > 0 \mid \ (\alpha_0 - \varepsilon_0)D_0 + \alpha D_1 \text{ is KLT at neither } x \text{ nor } x'\}$$

respectively. The definition of $\alpha_1$ is the same as in Case 1.1. But we note that $(\alpha_0 - \varepsilon_0)D_0$ is already not KLT at $x'$. We shall estimate $\alpha_1$. The proof of the following lemma is similar to that of Lemma 3.3.

**Lemma 3.7** *Let $\delta$ be the fixed positive number as above. Then we may assume that*

$$\alpha_1 \leqq \frac{n_1}{\sqrt[n_1]{\mu_1}} + \delta$$

*holds, if we take $\varepsilon'$, $\ell_1/m_1$ and $\varepsilon_0$ sufficiently small.* $\square$

This estimate is better than Lemma 3.3. Then we may define the proper subvariety $X_2$ of $X_1$ as the minimal center of log canonical singularities of $(X, (\alpha_0 - \varepsilon_0)D_0 + \alpha_1 D_1)$ at $x$ or $x'$ as we have defined $X_1$.

Lastly in Case 1.2 the construction of the filtration reduces to Case 2 as follows. In Case 1.2, $X_1$ does not pass through $x'$. Hence in this case the minimal center of LC singularities $X'_1$ at $x'$ does not pass through $x$. One may

reduce Case 1.2 to Case 2, by "strengthening" the singularity of $D_0$ along $X_1'$ as follows.

Let $a_1$ be a sufficiently large positive integer such that

$$H^0(X, \mathcal{O}_X(a_1K_X) \otimes \mathcal{I}_{X_1'}) \neq 0.$$

Let $\tau'$ be a general nonzero section of $H^0(X, \mathcal{O}_X(a_1K_X) \otimes \mathcal{I}_{X_1'})$. We note that there exists an effective $\mathbf{Q}$-divisor $G$ on $X$ such that $K_X - G$ is ample and $x$ is not contained in $\operatorname{Supp} G$ as we have seen before. Hence if we take $a_1$ sufficiently large, we may assume that the divisor $(\tau')$ does not contain $x$. In this case instead of $\sigma_0$, we shall use $\sigma_0^e \otimes \tau'$, taking a positive integer $e$ large. Let $D_0' := (m_0e + a_1)^{-1}(\sigma_0^e \otimes \tau')$.Let us define a positive rational number $\alpha_0'$ for $(X, D_0')$ similar to $\alpha_0$. Then by the construction of $\tau'$, then the minimal center of LC singularities of $(X, \alpha_0'D_0')$ at $x$ is $X_1$ and $(X, \alpha_0'D_0')$ is not LC at $x'$. Also we can make $\alpha_0'$ arbitrary close to $\alpha_0$ by taking $e$ sufficiently large. Hence we may assume that $\alpha_0'$ satisfies the same estimate :

$$\alpha_0' \leqq \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}} + \delta$$

as $\alpha_0$. And we may continue the construction of the filtration. In this way we can reduce Case 1.2 to Case 2.

In any case we construct the next stratum $X_2$ as the minimal center of log canonical singularities of $(X, (\alpha_0 - \varepsilon_0)D_0 + \alpha_1 D_1)$ at $x$. If $X_2$ is a point, then we stop the construction of the filtration. If $X_2$ is not a point, we continue exactly the same procedure replacing $X_1$ by $X_2$. And we continue the procedure as long as the new center of log canonical singularities $(X_1, X_2, \cdots)$ is not a point. As a result, for any distinct points $x, x' \in X^\circ$, inductively we construct a strictly decreasing sequence of subvarieties

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = x \text{ or } x'$$

and invariants :

$$\alpha_0, \alpha_1, \ldots, \alpha_r,$$

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-1},$$

$$n > n_1 > \cdots > n_r \qquad (n_i = \dim X_i, i = 1, \cdots, r),$$

and

$$\mu_0, \mu_1, \ldots, \mu_r \qquad (\mu_i := \mu(X_i, (K_X, h) \mid_{X_i}))$$

depending on small positive rational numbers $\varepsilon_0, \ldots, \varepsilon_{r-1}$, large positive integers $m_0, m_1, \ldots, m_r$, positive integers $0 =: \ell_0, \ell_1, \cdots, \ell_r$,

$$\sigma_i \in H^0(X, \mathcal{O}_X((m_i + \ell_i)K_X)) \qquad (i = 0, \cdots, r),$$

$$D_i = \frac{1}{m_i + \ell_i}(\sigma_i) \qquad (i = 0, \cdots, r),$$

etc.

By Nadel's vanishing theorem ([20, p.561]) we have the following lemma.

**Lemma 3.8** *For every positive integer $m > 1 + \sum_{i=0}^{r} \alpha_i$, $\Phi_{|mK_X|}$ separates $x$ and $x'$. And we may assume that*

$$\alpha_i \leqq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + \delta$$

*holds for every $0 \leqq i \leqq r$.* $\square$

**Proof**. For $i = 0, 1, \ldots, r$, let $h_i$ be the singular hermitian metric on $K_X$ defined by

$$h_i := \frac{1}{\mid \sigma_i \mid^{\frac{2}{m_i + \ell_i}}},$$

where we have set $\ell_0 = 0$. Using Kodaira's lemma ([14, Appendix]), let us take an effective $\mathbf{Q}$-divisor $G$ on $X$ such that $K_X - G$ is ample as before. As before we may assume that $\mathrm{Supp}\, G$ contains neither $x$ nor $x'$. Let $h'_G$ be a $C^\infty$-hermitian metric on the $\mathbf{Q}$-line bundle $K_X - G$ with strictly positive curvature. Let $G = \sum_k g_k G_k$ be the irreducible decomposition of $G$ and let $\sigma_{G_k}$ be a global holomorphic section of $\mathcal{O}_X(G_k)$ with divisor $G_k$. Then

$$h_G := h'_G \cdot (\prod_k \frac{1}{\mid \sigma_{G_k} \mid^{2g_k}})$$

is a singular hermitian metric of $K_X$ with strictly positive curvature current.

Let $m$ be a positive integer such that $m > 1 + \sum_{i=0}^{r} \alpha_i$ as above. Let $\varepsilon_G$ be a positive number such that

$$\varepsilon_G < m - 1 - (\sum_{i=0}^{r-1} (\alpha_i - \varepsilon_i) + \alpha_r).$$

We set

$$\beta := \sum_{i=0}^{r-1} (\alpha_i - \varepsilon_i) + \alpha_r + \varepsilon_G.$$

$$h_{x,x'} = (\prod_{i=0}^{r-1} h_i^{\alpha_i - \varepsilon_i}) \cdot h_r^{\alpha_r} \cdot h^{m-1-\beta} \cdot h_G^{\varepsilon_G}.$$

Then we see that $\mathcal{I}(h_{x,x'})$ defines a subscheme of $X$ with isolated support around $x$ or $x'$ by the definition of the invariants $\{\alpha_i\}$'s and the fact that $\mathrm{Supp}\, G$ contains neither $x$ nor $x'$. By the construction the curvature current $\Theta_{h_{x,x'}}$ is strictly positive on $X$. Then by Nadel's vanishing theorem ([20, p.561]) we see that

$$H^1(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h_{x,x'})) = 0$$

holds. Hence

$$H^0(X, \mathcal{O}_X(mK_X)) \to H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_X / \mathcal{I}(h_{x,x'}))$$

is surjective. Since by the construction of $h_{x,x'}$, $\mathrm{Supp}(\mathcal{O}_X/\mathcal{I}(h_{x,x'}))$ contains both $x$ and $x'$ and is isolated at least at one of $x$ or $x'$. Hence by the above surjection, there exists a section $\sigma \in H^0(X, \mathcal{O}_X(mK_X))$ such that

$$\sigma(x) \neq 0, \sigma(x') = 0$$

or

$$\sigma(x) = 0, \sigma(x') \neq 0$$

holds. This implies that $\Phi_{|mK_X|}$ separates $x$ and $x'$. The proof of the last statement is similar to that of Lemma 3.3. $\square$.

## 3.2 Estimate of the degree

To relate $\mu_0$ and the degree of pluricanonical images of $X$, we need the following lemma.

**Lemma 3.9** *If $\Phi_{|mK_X|}$ is a birational rational map onto its image, then*

$$\deg \Phi_{|mK_X|}(X) \leqq \mu_0 \cdot m^n$$

*holds.* $\square$

**Proof**. Let $p : \tilde{X} \longrightarrow X$ be the resolution of the base locus of $\mid mK_X \mid$ and let

$$p^* \mid mK_X \mid = \mid P_m \mid + F_m$$

be the decomposition into the free part $\mid P_m \mid$ and the fixed component $F_m$. We have

$$\deg \Phi_{|mK_X|}(X) = P_m^n$$

holds. Then by the ring structure of $R(X, K_X)$, we have an injection

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu P_m)) \to H^0(X, \mathcal{O}_X(m\nu K_X) \otimes \mathcal{I}(h^{m\nu}))$$

for every $\nu \geqq 1$, since the righthand side is isomorphic to $H^0(X, \mathcal{O}_X(m\nu K_X))$ by the definition of an AZD. We note that since $\mathcal{O}_{\tilde{X}}(\nu P_m)$ is globally generated on $\tilde{X}$, for every $\nu \geqq 1$ we have the injection

$$\mathcal{O}_{\tilde{X}}(\nu P_m) \to p^*(\mathcal{O}_X(m\nu K_X) \otimes \mathcal{I}(h^{m\nu})).$$

Hence there exists a natural homomorphism

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu P_m)) \to H^0(X, \mathcal{O}_X(m\nu K_X) \otimes \mathcal{I}(h^{m\nu}))$$

for every $\nu \geqq 1$. This homomorphism is clearly injective. This implies that

$$\mu_0 \geqq m^{-n} \cdot \mu(\tilde{X}, P_m)$$

holds by the definition of $\mu_0$. Since $P_m$ is nef and big on $X$, we see that

$$\mu(\tilde{X}, P_m) = P_m^n$$

holds. Hence

$$\mu_0 \geqq m^{-n} \cdot P_m^n$$

holds. This implies that

$$\deg \Phi_{|mK_X|}(X) \leqq \mu_0 \cdot m^n$$

holds. $\square$

### 3.3 Use of the subadjunction theorem

Let

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = x \text{ or } x'$$

be the filtration of $X$ as in Section 3.1.

**Lemma 3.10** *Let $W_j$ be a nonsingular model of $X_j$. For every $W_j$,*

$$\mu(W_j, K_{W_j}) \leqq (\lceil (1 + \sum_{i=0}^{j-1} \alpha_i) \rceil)^{n_j} \cdot \mu_j$$

*holds, where $\mu_j = (K_X, h)^{n_j} \cdot X_j$ as in Section 3.1 ( we note that $\mu(W_j, K_{W_j})$ depends only on $X_j$).* $\square$

**Proof**.

Let us set

$$\beta_j := \varepsilon_{j-1} + \sum_{i=0}^{j-1} (\alpha_i - \varepsilon_i).$$

Let $D_i$ denote the divisor $m_i^{-1}(\sigma_i)$ and we set

$$D := \sum_{i=1}^{j-1} (\alpha_i - \varepsilon_i) D_i + \varepsilon_{j-1} D_{j-1}.$$

Let $\pi : Y \longrightarrow X$ be a log resolution of $(X, D)$ which factors through an embedded resolution $\varpi : W_j \longrightarrow X_j$ of $X_j$. By the modification as in Section 3.1, we may assume that there exists a unique irreducible component $F_j$ of the exceptional divisor with discrepancy $-1$ which dominates $X_j$. Let

$$\pi_j : F_j \longrightarrow W_j$$

be the natural morphism induced by the construction. We set

$$\pi^*(K_X + D) \mid_{F_j} = K_{F_j} + G.$$

We may assume that the support of $G$ is a divisor with normal crossings. Then all the coefficients of the horizontal component $G^h$ of $G$ with respect to $\pi_j$ are less than 1 because $F_j$ is the unique exceptional divisor with discrepancy $-1$.

Let $dV$ be a $C^\infty$-volume form on the $X$. Let $\Psi$ be the function defined by

$$\Psi := \log(h^{\beta_j} \cdot \mid \sigma_{j-1} \mid^{\frac{2\varepsilon_{j-1}}{m_{j-1}}} \cdot \prod_{i=0}^{j-1} \mid \sigma_i \mid^{\frac{2(\alpha_i - \varepsilon_i)}{m_i}}).$$

Then the residue $\mathrm{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV))$ of $\pi^*(e^{-\Psi} \cdot dV)$ to $F_j$ is a singular volume form with algebraic singularities corresponding to the divisor $G$. Since every coefficient of $G^h$ is less than 1, there exists a nonempty Zariski open subset $W_j^0$ of $W_j$ such that $\mathrm{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV))$ is integrable on $\pi_j^{-1}(W_j^0)$.

Then the pullback of the residue $dV[\Psi]$ of $e^{-\Psi} \cdot dV$ (to $X_j$) to $W_j$ is given by the fiber integral of the above singular volume form $\mathrm{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV))$ on $F_j$, i.e.,

$$\varpi^* dV[\Psi] = \int_{F_j/W_j} \mathrm{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV))$$

holds. By Theorem 2.28, we see that $(K_{F_j} + G) - \pi_j^*(K_{W_j} + \Delta)$ is nef, where $\Delta$ is the $\mathbf{Q}$-divisor defined as in Theorem 2.28. We note that $K_{F_j} + G$ is $\mathbf{Q}$-linear equivalent to $(1 + \beta_j)\pi^* K_X$ by the construction. Hence we see that $(1 + \beta_j)\varpi^* K_X - (K_{W_j} + \Delta)$ is nef and

$$\mu(W_j, K_{W_j}) \leqq \mu(W_j, (1 + \beta_j)\varpi^*(K_X \mid_{X_j}) - \Delta) \tag{1}$$

holds.

Let $e$ be a positive integer such that $e \cdot \Delta$ is an integral divisor. Let $\sigma_{e\cdot\Delta}$ be a meromorphic section of $\mathcal{O}_{W_j}(e \cdot \Delta)$ with divisor $e \cdot \Delta$. Then we may consider the $e$-th root $\sigma_\Delta$ of $\sigma_{e\cdot\Delta}$ as a multivalued meromorphic section of the $\mathbf{Q}$-line bundle $\mathcal{O}_{W_j}(\Delta)$ with divisor $\Delta$. Let $h_\Delta$ be a $C^\infty$-hermitian metric on the $\mathbf{Q}$-line bundle $\mathcal{O}_{W_j}(\Delta)$, i.e., $h_\Delta$ is the $e$-th root of a $C^\infty$-hermitian metric on the line bundle $\mathcal{O}_{W_j}(e \cdot \Delta)$. Then $h_\Delta(\sigma_\Delta, \sigma_\Delta)$ is a single valued funtion on $W_j$.

Let us recall the interpretation of the divisor $\Delta$ in Section 3.7. Let $dV_{W_j}$ be a $C^\infty$-volume form on $W_j$. We note that in the above definition of the function $\Psi$, we have used $h^{\beta_j}$ instead of $dV^{-\beta_j}$. Hence we see that there exists a positive constant $C$ such that

$$\varpi^* dV[\Psi] = \int_{F_j/W_j} \mathrm{Res}_{F_j}(\pi^*(e^{-\Psi} \cdot dV)) \leqq C \cdot \frac{\varpi^*(dV \cdot h)^{-\beta_j}}{h_\Delta(\sigma_\Delta, \sigma_\Delta)} \cdot dV_{W_j} \tag{2}$$

hold.

We may assume that $\beta_j$ is not an integer without loss of generality. In fact this can be satisfied, if we perturb $\varepsilon_0, \cdots, \varepsilon_{j-1}$ or $\sigma_0, \cdots, \sigma_{j-1}$. And passing to the lmit, the general case follows. This condition is to assure the inequality $\lceil 1 + \beta_j \rceil > 1 + \beta_j$ and this inequality corresponds to the condition : $d > \alpha m_0$ in Theorem 2.22. We note that for every positive integer $m$, every global holomorphic section of $mK_X$ is bounded with respect to $h^m$. Then since the curvature current $\Theta_h$ is semipositive in the sense of current, applying Theorem 2.22 (see also Remark 2.26 for the selfcontainedness) , we have the interpolation :

$$\begin{aligned} A^2(W_j, m(\lceil 1 + \beta_j \rceil)\varpi^* K_X, \varpi^*(e^{-(m-1)\varphi} \cdot dV^{-m} \otimes h^{m\lceil \beta_j \rceil}), \varpi^* dV[\Psi]) \\ \to H^0(X, \mathcal{O}_X(m(\lceil 1 + \beta_j \rceil)K_X)), \end{aligned}$$

where $\varphi$ is the weight function defined by

$$\varphi := \log \frac{dV_{W_j}}{\varpi^* dV[\Psi]}$$

as in Theorem 2.22. By (2), we see that

$$\varpi^*(e^{-\varphi} \cdot dV^{-1} \otimes h^{\beta_j} \mid_{X_j}) \leqq C \cdot h_\Delta(\sigma_\Delta, \sigma_\Delta)^{-1} \cdot \varpi^*(dV^{-(1+\beta_j)} \mid_{X_j}) \tag{3}$$

holds. We note that $\Delta$ may not be effective. Hence a priori the element of $A^2(W_j, m(\lceil 1 + \beta_j \rceil)\varpi^* K_X, \varpi^*(e^{-(m-1)\varphi} \cdot dV^{-m} \otimes h^{m\lceil \beta_j \rceil}), \varpi^* dV[\Psi])$ may have pole along the degenerate locus (zero locus) of $\varpi^* dV[\Psi]$. But this cannot occur by the existence of the extension and the birational invariance of plurigenera. As in the remark below, we also may reduce the proof to the case that $\Delta$ is effective.

Since $(1+\beta_j)\varpi^*(K_X\mid_{X_j}) - (K_{W_j}+\Delta)$ is nef (This is nothing but the main part of the proof of Kawamata's subadjunction theorem [11, Theorem 1]. Then the proof of [11, Theorem 1] follows from the observation that $\varpi_*\Delta$ is effective), by using Theorem 2.28, noting the equality $dV[\Psi] = e^{-\varphi}\cdot dV_{W_j}$, the inequalities (1),(3) and the existence of the above interpolation imply that

$$\mu(W_j, K_{W_j}) \leqq$$
$$n_j! \cdot \overline{\lim}_{m\to\infty} m^{-n_j} \dim \mathrm{Image}\,\{H^0(X, \mathcal{O}_X(m(\lceil 1+\beta_j\rceil)K_X)) \to H^0(X_j, \mathcal{O}_{X_j}(m(\lceil 1+\beta_j\rceil)K_X))\}$$

holds. Here we have used the fact that for any pseudoeffective divisors $M_1, M_2$ on a smooth projective variety $V$ such that $M_1 - M_2$ is pseudoeffective, the inequality: $\mu(V, M_1) \geqq \mu(V, M_2)$ holds (the proof is trivial and left to the reader).

Since every element of $H^0(X, \mathcal{O}_X(m(\lceil 1+\beta_j\rceil)K_X))$ is bounded on $X$ with respect to $h^{m(\lceil 1+\beta_j\rceil)}$ (cf. Remark 2.14). In particular the restriction of an element of $H^0(X, \mathcal{O}_X(m(\lceil 1+\beta_j\rceil)K_X))$ to $X_j$ is bounded with respect to $h^{m(\lceil 1+\beta_j\rceil)}\mid_{X_j}$. Hence by the existence of the above interpolation, we have that

$$\mu(W_j, K_{W_j}) \leqq \mu(X_j, (\lceil 1+\beta_j\rceil)K_X, h^{\lceil 1+\beta_j\rceil})\mid_{X_j}) \tag{4}$$

holds. This is the only point where Theorem 2.22 is used.

By the trivial inequality

$$\beta_j \leqq \sum_{i=0}^{j-1}\alpha_i.$$

we have that

$$\mu(W_j, K_{W_j}) \leqq (\lceil 1 + \sum_{i=0}^{j-1}\alpha_i\rceil)^{n_j}(K_X, h)^{n_j}\cdot X_j$$

holds by the definition of $(K_X, h)^{n_j}\cdot X_j$. This is the desired inequality, since $\mu_j = (K_X, h)^{n_j}\cdot X_j$ holds by the definition of $\mu_j$. $\square$

**Remark 3.11** *In the above proof, the divisor $\Delta$ on $W_j$ may not be effective. But it is clear that $\varpi_*\Delta$ is effective (cf. the proof of [11, Theorem 1]). If we replace $X_j$ by $W_j$ and $X$ by the the ambient space of the embedded resolution $\varpi : W_j \longrightarrow X_j$, we may reduce the above proof to the case that $X_j$ is already smooth. In this case we may assume that $\Delta$ is effective.*

Now we shall complete the proofs of Theorems 1.1 and 1.2.

Suppose that Theorem 1.2 holds for every projective varieties of general type of dimension $< n$, i.e., there exist positive constants $\{C(k)(k<n)\}$ such that for every smooth projective $k$-fold $Y$ of general type

$$\mu(Y, K_Y) \geqq C(k)$$

holds. Let $X$ be a smooth projective variety of general type of dimension $n$. Let $U_0$ be a nonempty open subset of $X$ with respect to **countable Zariski topology** such that for every $x \in U_0$ there exist no subvarieties of nongeneral type containing $x$. Such a set $U_0$ surely exists, since there exists no dominant family of subvarieties of nongeneral type in $X$. In fact if such a dominant family exists, then this contradicts the assumption that $X$ is of general type. Then if

$(x,x') \in (U_0 \times U_0)\backslash\Delta_X$, the stratum $X_j$ as in Section 3.1 is of general type for every $j$ by the definition of $U_0$. By Lemma 3.10 and the definition of $C(n_j)$, we see that

$$C(n_j) \leqq (\lceil(1 + \sum_{i=0}^{j-1}\alpha_i)\rceil)^{n_j} \cdot \mu_j \tag{5}$$

holds for $W_j$. Since

$$\alpha_i \leqq \frac{\sqrt[n_i]{2}\, n_i}{\sqrt[n_i]{\mu_i}} + \delta \tag{6}$$

holds for every $0 \leqq i \leqq r$ by Lemma 3.8, combining (5) and (6), we see that

$$\frac{1}{\sqrt[n_j]{\mu_j}} \leqq (2 + \sum_{i=0}^{j-1}\frac{\sqrt[n_i]{2}\, n_i}{\sqrt[n_i]{\mu_i}}) \cdot C(n_j)^{-\frac{1}{n_j}}$$

holds for every $j \geqq 1$.

Using the above inequality inductively, we obtain the following lemma.

**Lemma 3.12** *Suppose that $\mu_0 \leqq 1$ holds. Then there exists a positive constant $C$ depending only on $n$ such that for every $(x,x') \in (U_0 \times U_0)\backslash\Delta_X$ the corresponding invariants $\{\mu_0, \cdots, \mu_r\}$ and $\{n_1, \cdots, n_r\}$ depending on $(x,x')$ ($r$ may also depend on $(x,x')$) satisfies the inequality :*

$$2 + \lceil\sum_{i=0}^{r}\frac{\sqrt[n_i]{2}\, n_i}{\sqrt[n_i]{\mu_i}}\rceil \leqq \lfloor\frac{C}{\sqrt[n]{\mu_0}}\rfloor.$$

□

We note that $\{n_1, \cdots, n_r\}$ is a strictly decreasing sequence and this sequence has only finitely many possibilities. By Lemmas 3.8 and 3.11 we see that for

$$m := \lfloor\frac{C}{\sqrt[n]{\mu_0}}\rfloor,$$

$\mid mK_X \mid$ separates points on $U_0$. Hence $\mid mK_X \mid$ gives a birational embedding of $X$.

Then by Lemma 3.9, if $\mu_0 \leqq 1$ holds,

$$\deg \Phi_{|mK_X|}(X) \leqq C^n$$

holds. Also

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \leqq n + 1 + \deg \Phi_{|mK_X|}(X)$$

holds by the semipositivity of the $\Delta$-genus ([7]). Hence we have that if $\mu_0 \leqq 1$,

$$\dim H^0(X, \mathcal{O}_X(mK_X)) \leqq n + 1 + C^n$$

holds.

Since $C$ is a positive constant depending only on $n$, combining the above two inequalities, we have that there exists a positive constant $C(n)$ depending only on $n$ such that

$$\mu_0 \geqq C(n)$$

holds.

More precisely we argue as follows. Let $\mathcal{H}$ be the union of the irreducible components of the Hilbert scheme of projective spaces of dimension $\leqq n + C^n$ and the degree $\leqq C^n$. By the general theory of Hilbert schemes ([8, exposé 221]), $\mathcal{H}$ consists of finitely many irreducible components. Let $\mathcal{H}_0$ be the Zariski open subset of $\mathcal{H}$ which parametrizes irreducible subvarieties. Then there exists a finite stratification of $\mathcal{H}_0$ by Zariski locally closed subsets such that on each stratum, there exists a simultaneous resolution of the universal family on the stratum. We note that the volume of the canonical bundle of the resolution is constant on each stratum by [31, 21]. Hence there exists a positive constant $C(n)$ depending only on $n$ such that

$$\mu(X, K_X) \geqq C(n)$$

holds for every projective $n$-fold $X$ of general type by the degree bound as above. This completes the proof of Theorem 1.2. $\square$

Now let us prove Theorem 1.1. Then by Lemmas 3.8 and 3.11, we see that there exists a positive integer $\nu_n$ depending only on $n$ such that for every projective $n$-fold $X$ of general type, $\mid mK_X \mid$ gives a birational embedding into a projective space for every $m \geqq \nu_n$. This completes the proof of Theorem 1.1. $\square$

# 4 The Severi-Iitaka conjecture

Let $X$ be a smooth projective variety. We set

$$Sev(X) := \{(f, [Y]) \mid f : X \longrightarrow Y \ \text{ dominant rational map and } Y \text{ is of general type}\},$$

where $[Y]$ denotes the birational class of $Y$. By Theorem 1.1 and [18, p.117, Proposition 6.5] we obtain the following theorem.

**Theorem 4.1** *$Sev(X)$ is finite.* $\square$

**Remark 4.2** *In the case of $\dim Y = 1$, Theorem 4.1 is known as Severi's theorem. In the case of $\dim Y = 2$, Theorem 4.1 has already been known by K. Maehara ([18]). In the case of $\dim Y = 3$, Theorem 4.1 has recently proved by T. Bandman and G. Dethloff ([2]).* $\square$

# References


[1] U. Anghern-Y.-T. Siu, Effective freeness and point separation for adjoint bundles, Invent. Math. 122 (1995), 291-308.

[2] T. Bandeman- G. Detholoff, Estimates on the number of rational maps from a fixed variety to varieties of general type, Ann. Inst. Fourier 47, 801-824 (1997).

[3] E. Bombieri, Canonical models of surfaces of general type, Publ. I.H.E.S. 42 (1972), 171-219.

[4] J.P. Demailly, A numerical criterion for very ample line bundles, J. Diff. Geom. 37 (1993), 323-374.

[5] J.P. Demailly, Regularization of closed positive currents and intersection theory, J. of Alg. Geom. 1 (1992) 361-409.

[6] J.P. Demailly-T. Peternell-M. Schneider : Pseudo-effective line bundles on compact Kähler manifolds, math. AG/0006025 (2000).

[7] T. Fujita, On the structure of polarized varieties with $\Delta$-genera zero. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), 103–115.

[8] A. Grothendiek, Fondements de la géometrie algébrique, collected Bourbaki talks, Paris (1962).

[9] R. Harthshorne, Algebraic geometry, GTM 52, Springer (1977).

[10] L. Hörmander, An Introduction to Complex Analysis in Several Variables 3-rd ed.,North-Holland(1990).

[11] Y. Kawamata, Subadjunction of log canonical divisors II, alg-geom math.AG/9712014, Amer. J. of Math. 120 (1998),893-899.

[12] Y. Kawamata, Fujita's freeness conjecture for 3-folds and 4-folds, Math. Ann. 308 (1997), 491-505.

[13] Y. Kawamata, Semipositivity, vanishing and applications, Lecture note in School ON VANISHING THEOREMS AND EFFECTIVE RESULTS IN ALGEBRAIC GEOMETRY (ICTP, Trieste, May 2000).

[14] S. Kobayashi-T. Ochiai, Mappings into compact complex manifolds with negative first Chern class, Jour. Math. Soc. Japan 23 (1971),137-148.

[15] J. Kollár- S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Math., Cambridge University Press (1998).

[16] S. Krantz, Function theory of several complex variables, John Wiley and Sons (1982).

[17] P. Lelong, Fonctions Plurisousharmoniques et Formes Differentielles Positives, Gordon and Breach (1968).

[18] K. Maehara, A finiteness properties of varieties of general type, Math Ann. 262 (1983),101-123.

[19] S. Mori, Flip conjecture and the existence of minimal model for 3-folds, J. of A.M.S. 1 (1988), 117-253.

[20] A.M. Nadel, Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. 132 (1990),549-596.

[21] N. Nakayama, Invariance of plurigenera of algebraic varieties, RIMS preprint 1191, March (1998).

[22] T. Ohsawa and K. Takegoshi, $L^2$-extension of holomorphic functions, Math. Z. 195 (1987),197-204.

[23] T. Ohsawa, On the extension of $L^2$ holomorphic functions V, effects of generalization, Nagoya Math. J. (2001) 1-21.

[24] Y.-T. Siu, Invariance of plurigenera, Invent. Math. 134 (1998), 661-673.

[25] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, Jour. Diff. Geom. 32 (1990),99-130.

[26] H. Tsuji, Analytic Zariski decomposition, Proc. of Japan Acad. 61(1992) 161-163.

[27] H. Tsuji, Existence and Applications of Analytic Zariski Decompositions, Trends in Math. Analysis and Geometry in Several Complex Variables, (1999) 253-272.

[28] H. Tsuji, Numeircally trivial fibrations, math.AG/0001023(2000).

[29] H. Tsuji, On the structure of pluricanonical systems of projective varieties of general type, preprint (1997).

[30] H. Tsuji, Global generation of adjoint bundles, Nagoya Math. J. 142 (1996),5-16.

[31] H. Tsuji, Deformation invariance of plurigenera, Nagoya Math. J. 166 (2002), 117-134.

[32] H. Tsuji, Subadjunction theorem, Advanced Studies in Pure Math. 42, (2004), 313-318.

[33] H. Tsuji, Pluricanonical systems of projective varieties of general type, math.AG/9909021 (1999).

[34] H. Tsuji, Pluricanonical systems of projective varieties of general type I (2004).

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